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AUTOMATIC GENERATION OF SOLID MODELS FROM 3D SEISMIC INTERPRETATIONS

by

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1. INTRODUCTION

This paper decribes a procedure for automatic generation of solid modelling computer representations of the subsurface from 3-dimensional seismic interpretations.

The result of a 3D seismic interpretation is usually some representation of the *horizons* in the area. The layers or blocks of rock which actually constitute the subsurface, are not explicitly mapped. However, detailed analysis of complex structures may be greatly facilitated if also the layers and blocks can be handled directly, i.e. if a *solid modelling* representation of the subsurface is available.

A solid representation is advantageous, f.ex., in depth conversion, modelling of the geological evolution, seismic and gravimetric modelling and reservoir studies. In contrast to maps and cross sections, a solid model representation allows the user to handle directly the 3-dimensional solid blocks in the model: combine, split and move blocks, assign rock properties to the blocks, calculate volume and mass etc.

A solid modelling representation consists of two parts: firstly, a representation of the horizons, or interfaces, in the model, and secondly some logical system describing the solids in terms of the interfaces. The choice of representation depends on model complexity, characteristics of the interfaces, computer speed/memory considerations etc. Solid modelling representations have primarily been used in computer-aided design and manufacturing (CAD/CAM) (see f.ex. the article of Requicha and Voelcker (1982)). The use of solid modelling in geophysics has been scarce, but the method developed by Gjøystdal et al. (1983) is designed specifically for exploration geophysics applications. There are some important differences between typical models frequently used in CAD/CAM and in exploration geophysics: The interfaces may have very general shapes in geophysical models, while in CAD/CAM mostly simple, analytical interfaces are used. On the other hand, model complexity, in terms of number of interfaces and volumes, is usually larger in CAD/CAM.

From a 3D seismic interpretation the interfaces are readily available. Thus, if the available interface representation can be used in the application in question, what remains to be made is the logic describing the volumes in terms of the interfaces. The logical set-up may e.g. be specified manually following the principles of Gjøystdal et al. (1983). However, when certain conditions are fulfilled, a logical volume description based on the same principles can be set up automatically.

The paper consists of two parts. The first part describes the principles of automatic solid model generation. In the second part, practical aspects related to the implementation of the procedure on a computer are discussed.

2. PRINCIPLES

2.1 General remarks

The automatic solid model generator makes a 3D solid model from a set of interfaces, which e.g. may be the result of a seismic interpretation. The interfaces may be depth interfaces or migrated two-way travel time interfaces. If the latter are used, depth conversion by image ray-tracing (Gjøystdal and Åstebøl,1983) is necessary in order to convert the model to the depth domain. Depth conversion can easily be done after the solid model has been made if interval velocities are available.

To ensure a proper volume determination, the set of interfaces must fulfill certain requirements which are discussed in Section 2.2. The model generation consists of two steps (Sections 2.3 and 2.4). First a 'volume matrix' is made from a detailed analysis of the interfaces, in particular of their intersection lines. The volume matrix is a gridded, preliminary solid representation. Then, by combined inspection of the volume matrix and the interfaces, the logical representations of the volumes in terms of the interfaces are determined.

The model is limited by a 3D rectangular box (model box), which is completely filled with non-intersecting blocks.

2.2 Requirements to the interfaces

The assumptions used in the volume determination are:
- Each interface is continuous and fairly smooth.
- Each volume is completely enclosed by interfaces and/or the model box.
- An interface cannot be partly above and partly below the same volume.
- An interface is sampled only where it is the physical boundary between two different volumes.
- The sampling is dense relative to the model complexity.

Important consequences of the above assumptions are:
- If an interface is discontinuous, e.g. due to faulting, or if it is not smooth, the continuous, smooth parts should be separated and considered as different interfaces.
- Every point in the model must be inside one and only one volume.
- All boundaries between blocks must be sampled.
- Interfaces may not intrude the volumes, that is, interfaces must terminate towards other interfaces or at the model boundary.

Thus the seismic interpretation must be made with the 'solid concept' in mind. Simple examples of valid and invalid interface sets are shown in Fig.1.

Furthermore, all interfaces must be on the form z=f(x,y), i.e. the time/depth must be a function of the horizontal coordinates only.

This restriction prohibits, e.g., vertical and near-vertical fault planes and very steep folds and domes.

To simplify the analysis it is also assumed that all interfaces are sampled on the same, regular, horizontal xy-grid. As this representation is common in computer assisted interpretation, solid modelling is specially simple when the 3D seismic data are interpreted on an interactive interpretation station. If the interfaces are not represented on this form, they should be (re)sampled prior to the solid model generation.

If the requirements are not fulfilled, unexpected - and unwanted - volumes will result, leading to an erroneous solid model.

2.3 First step - build a volume matrix

The interfaces are sampled on a regular grid (Xi,Yj), i=1,N j=1,M where Xi = X1 + (i-1) * dX and Yi= Y1 + (j-1) * dY. dX and dY are the grid spacings in x- and y-direction.

The set of interfaces is represented by a 3-dimensional 'z-matrix'
z(i,j,k) i=1,N j=1,M k=1,K
where z(i,j,k) is the depth or time value for the interface with index k in grid point i,j. K is the number of interfaces in the model.
If interface k is not defined in (i,j), it is flagged by setting a special value of z(i,j,k). To simplify the analysis, two horizontal interfaces are always automatically included in the z-matrix, one at the top of the model and one at the bottom. These two interfaces are defined in all grid points.

Sorting the z-values in each grid point with increasing depth, an 'interface matrix' is easily made:

$k(i,j,l)$ $i=1,N$ $j=1,M$ $l=1,L(i,j)$

$k(i,j,l)$ is the interface index of the l-th interface counted from the top of the model in (i,j). $L(i,j)$ is the number of interfaces defined in (i,j). Thus $k(i,j,l)$ contains the ordered sequence of interfaces for each grid point, arranged with increasing depth/time.

The task is now to make a 'volume matrix'

$v(i,j,l)$ $i=1,N$ $j=1,M$ $l=1,L(i,j)-1$

where $v(i,j,l)$ is the index of the volume right below interface $k(i,j,l)$ in (i,j). Thus $v(i,j,l)$ contains the ordered sequence of volumes for each grid point, arranged with increasing depth/time. The number of volumes in each grid point is one less than the number of interfaces, as no volume is defined above the top interface nor below the bottom interface.

To determine the volume matrix, the z- and interface matrices are investigated systematically grid point by grid point, and the same procedure is in principle used in all points, except the first one, to determine the sequence of volume indices.

In the first point (1,1) nothing is yet known about the volumes, so they are simply numbered sequentially, starting with volume index 1 under the top interface, index 2 under the next one, etc.

In the other points the procedure is as follows:

Assume that the volume determination has been made rowwise from (1,1) and has been successful for all points up to and including (i_1,j_1-1). The next point to be analysed ('current point') is (i_1,j_1).

To determine the volume sequence in (i_1,j_1), the interface sequence $k(i_1,j_1,l)$ $l=1,L(i_1,j_1)$ is compared with the similar sequences in up to four neighbouring, already investigated points (Fig.2). Each of the neighbouring points is treated separately, thus reducing the volume determination to a 2-dimensional problem. The independently established volume sequences for (i_1,j_1) are then compared and checked for differences.

The two-point comparison procedure takes advantage of the fact that the interfaces are sampled exactly where and only where they are physical boundaries, and of the dense sampling relative to changes in the model. For a model with these properties, it is reasonable to assume that within the range of a few grid points, the interfaces are well approximated by planes. It can also be assumed that in a vertical plane through two neighbouring grid points, each interface intersects with at most one interface. On the above presumptions, only a few different patterns of interface intersections may occur, and the unknown volume sequence is inferred from a few, simple rules. The rules are illustrated in Fig.3 and described in the caption.

In a complex 3D model complex regions may occur where the assumptions are not valid for all combinations of (i_1,j_1) and neighbouring points. Still, the volume determination is usually resolved, as in most cases the volume sequence is sufficiently determined by one or two two-point comparisons. In most cases, the use of four neighbouring points 'overdetermines' the volume sequence. Only with some rare, very complex geometries, manual interaction is necessary to establish the volume sequence.

Though the volume sequence for (i_1,j_1) is properly determined in different two-point comparisons, the resultant sequences may sometimes be different. Inconsistencies occur because different parts of a volume may appear as separate volumes as long as only a part of the model has been inspected. The different two-point comparisons use information from different parts of the model and might therefore be 'cheated'.

The inconsistencies are resolved by renumbering of volumes. This comparison of different volume sequences for the same point is a very important part of the volume determination procedure, as it represents the 'global overview' in the process (Fig.4).

2.4 Second step - determine logical volume representations

The volume, interface and z-matrices comprise a solid modelling representation. However, a second step is introduced to convert the volumes to the more powerful representation outlined in Gjøystdal et al. (1983). The reader should be familiar with the principles outlined in the paper. With this volume logic, the model is easily modified by means of Boolean operations on the volumes. Also more general volumes can be made, in that interfaces represented in different coordinate systems (cartesian, sylindrical, spherical) can be combined with the original model.

So far, the interfaces have only been defined in the grid points, and their borders have been located by means of linear extrapolation. In most applications, values outside the grid points are needed, and the interfaces must in some cases fulfil strict continuity requirements. This is the case in dynamic ray-tracing, where the z-values must be continuous up to the second partial derivatives. For this reason, spline approximations to the interfaces are used.

The matrix independent volume logic is very flexible with respect to interface representations. Any representation can be used which provides z-values in all points inside the 'areas of definition'. For each interface-volume combination in the model, a separate area of definition is determined. The area of definition is derived from the z-matrix and covers the entire area where the interface is a boundary of the volume.

Detailed sampling of the interface intersection curves is inconvenient, so the volume logic allows areas of definition stretched slightly outside these curves. However, when two interfaces on opposite sides of a volume intersect, the 'non-physical' extensions of their areas of definition are made coincident. For practical reasons, an area of definition consists of a set of rectangles on a regular grid, which does not have to be the original z-matrix grid. (Fig.5).

The logical volume build-up is based on the same ideas as polygon fill on graphic raster screens (see e.g. Foley and Van Dam (1982)), but is generalized to 3D space.

The logical representation V of a volume v is set up following these rules ('=' means compute right hand side and asssign to the left hand side):

a) Start with $V = \emptyset$ (no volume).

b) Among the interfaces bounding V, select the uppermost, previously 'unused' interface(s). More than one is considered as 'the uppermost' if there are laterally non-overlapping interfaces. For each of these interfaces k_j define the 'Simple Block' $SBk_j = SB(+k_j)$. This is the part of the 3D space which is below k_j and is limited laterally by the area of definition for k_j combined with V. The SBs are non-physical volumes used in the logical represention of the real volumes. For more details on SBs see Gjøystdal et al. (1983).

   Check if k_j intersects with other interfaces which lie on the same side of the volume. Whenever such intersection occurs, and the non-physical continuation of one of the interfaces is **above** the physical part of the other one, two additional SBs must be made. Say k_j intersects with k_i, then the SBs $SB1 = SB(+k_j,-k_i)$ and $SB2 = SB(-k_j,+k_i)$ are made. $SB(+k_j,-k_i)$ consists of all points that are below k_j and above k_i and are inside the two areas of definition for the interfaces combined with V.

The two SBs are subtracted from SBk_j :

$$SBk_j = SBk_j - (SB1 \cup SB2)$$

The subtraction is repeated for all intersections of this type with k_j.

The next step is to combine the resultant SBk_j with the 'current volume' V:

If the interface k_j is a b o v e the volume, take the union of $SB(+k_j)$ and V :

$$V = V \cup SB(+k_j)$$

If the interface is b e l o w the volume, $SB(+k_j)$ is subtracted from V:

$$V = V - SB(+k_j)$$

The interface k_j is now considered as 'used'

c) Use b) repeatedly untill all interfaces have the status 'used'. The generation of logical volume representations is illustrated in Fig.6.

3. PROBLEMS RELATED TO COMPUTER IMPLEMENTATION

3.1 General remarks

The principles for automatic solid model generation as described in Chapter 2, are suited for computer implementation. Still, during the implementation and use of the model generation programs in the GNOM (GECO NORSAR MODELLING) package, some methodical problems revealed. They are discussed in some detail in the following, and solutions are suggested.

In the GNOM package, the automatic solid model generator is implemented with the modifications outlined in this chapter.

3.2 Correct sampling of the interfaces

The first problem is to ensure that the set of sampled interfaces fulfils the requirements of Section 2.2. This is done in two steps: First the interpreter must check that the model roughly is correct. Then an automatic 'preprocessor' of the interfaces removes smaller 'errors' in the model.

The interpreter must have in mind that a solid model shall be made. The horizons must be drawn so that they completely enclose the volumes, and so that no horizons intrude the volumes. Unfortunately, this is often in conflict with common practice in interpretation. Therefore, the interpreted horizons should be manually reexamined and corrected prior to model generation. This check should be done interactively, e.g. on an interpretation station. In particular it should be noted that each interface must be continuous. Often an interface in the interpretation consists of several completely separated parts which belong to the same geological horizon. Such interfaces should be divided and separate indices assigned to the different continuous parts.

If the interpreter has violated the restrictions set by the solid concept, and the interpretation is not corrected, large 'errors' in the solid model must be expected.

The volume matrix builder (Section 2.3) analyses the interfaces grid point by grid point and is therefore rather sensitive to 'errors' in the z-matrix on the sampling level. The critical assumption is that the interfaces shall be sampled where and only where they are physical volume boundaries. Combining this with the assumptions of continuous and smooth interfaces and dense sampling, it is possible to infer where samples are 'missing' and where extra, 'erroneous' samples have been added: Along the interface boundaries the interfaces are linearly extrapolated in the two grid directions (constant i and constant j), and intersection points between the interfaces are determined. For each interface boundary the nearest intersection point is located. If the point is on the interface, the samples between the intersection point and the boundary are removed. On the other hand, if the intersection point is outside the interface, and also outside the grid rectangle adjacent to the boundary sample, samples are added (Fig.7). The result of this resampling is usually very close to what intuitively would be considered as 'correct' in a manual inspection.

3.3 B-spline interface representation

The implemented automatic model generator has been specifically designed for 3D dynamic ray-tracing. In this application up to the second partial derivatives of the z-values must be continuous. This is fulfilled by B-splines. If the interface is smooth, the fit to the original interface data is good. Still unwanted effects may occur along the interface boundaries. There is 'data control' on one side only of the boundaries, and the B-spline tends to oscillate dramatically in such areas. In some cases this may distort the volumes (Fig.8). To cope with this problem, some samples are added preliminary along the interface rim during the B-spline generation. These samples are determined by linear extrapolation out from the defined part of the interface. In this manner the area with complete data control is extended to include the interface boundary regions.

The final, logical volume representation uses only the B-spline interfaces. Still, the volume matrix is made by analysis of the z-matrix, and the volume logic is determined from the volume and z-matrices. Therefore, the z-matrix and the B-spline interfaces must be mutually consistent. Due to the smoothing effect of the B-splines, there may be small differences between values in the z-matrix and the corresponding ones computed from the splines. In these cases the value in the z-matrix should be replaced by the spline value. The differences may also shift the interface intersection curves slightly so that a few samples must be added or removed. Still, the changes are usually small, and the original model is essentially retained.

3.4 Revised volume logic

In geophysical models, each interface is often the boundary of several volumes. With the procedure in Section 2.4, an area of definition is defined for each interface-volume combination. Conceptually, it is better to define a 'global' area of definiton for each interface, as this decouples the interface definitions completely from the volumes. This is particularly advantageous if the model shall be modified. The interface can be modified with no change in volume definition as long as the modification does not change the over/under relationships between the interfaces. With a separate area of definition for each interface-volume pair, all these areas of defintion must be reset if the lateral extensions of an interface is changed.

In addition, the use of global areas of definition saves space in the computer memory. This may be of importance for large models.

The disadvantage with global areas of definition is that the volume logic becomes more complex and less general.

With global areas of definition the following volume logic build-up procedure is used (Fig.9):

a) A volume V1 is made following the rules in Section 2.4.

b) A similar volume V2 is made, but now the procedure starts at the b o t t o m of the volume and proceeeds u p w a r d s.

c) A v o l u m e area of definition is defined. It is similar to the interface areas of definition, but is determined from the lateral extension of the volume.

d) A volume V3 is made, consisting of all points in 3D space that laterally are inside the volume area of definition.

e) The final volume V is the intersection of the three volumes: $V = V1 \cap V2 \cap V3$.

This logic is enable to represent all volumes that are likely to occur in geophysical models. However, it fails for some very complex, rather hypothetical volume geometries. As a guidance, an erroneous volume representation can only be made if there is a grid point where at least six interfaces bound the same volume. An example of erroneous volume representation is shown in Fig.10.

If models involve volumes of this complexity, the method of Section 2.4 should be applied.

Acknowledgements

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FIGURE CAPTIONS

Fig. 1

Figure 1 a) - c) show some typical invalid interface sets in automatic solid model generation and similar, valid sets.

a) An interface intrudes a volume or is not sampled along the entire volume boundary.

b) An interface stops in the middle of a volume.

c) An interface (number 5) is discontinuous. Different indices must be assigned to the separate, continuous parts.

Fig. 2

The volume sequence in (i_1, j_1) is determined by comparison of the interface sequence with the corresponding ones in up to four neighbouring points. In these points the volume sequences have already been established.

Fig. 3

Figure 3 a) - f) illustrates the rules applied in the 'two-point comparison' procedure to determine the volume sequence in a grid point. The volume sequence is known in (m,n) and shall be determined in (i,j). Plane, local approximations to the interfaces, computed from the z-matrix, are used in the comparison. In the illustrations, the planes are drawn between points where the interfaces are defined, and from a point where an interface is defined and to its intersection point with another interface. k_n denotes interface, v_n denotes volume.

a) If an interface k_1 is defined in (i,j) as well as in (m,n), and it is not intersected by any interface above k_1, the volume right above k_1 in (m,n) is also right above k_1 in (i,j).

b) If an interface k_2 is defined in (i,j) as well as in (m,n), and it is is intersected by an interface k_1 which is above k_2 in (m,n), the volume v_1 right above k_1 in (m,n) is right above k_2 in (i,j).

c) If an interface k_2 is defined in (i,j) as well as in (m,n), and it is is intersected by an interface k_1 which is above k_2 in (i,j), the volume v_1 right above k2 in (m,n) is right above k1 in (i,j). A 'new' volume (as far as can be inferred from this two-point comparison) occurs between k_1 and k_2 in (i,j).

d) If an interface k_1 which is defined in (m,n) only, intersects with k_2 which is defined in (i,j) only, the volume v_1 right above k_1 in (m,n) is right above k_2 in (i,j).

Obviously, similar arguments as in a) - d) apply for the area b e l o w the interfaces.

e) If an interface k_1 which is defined in (m,n) only, intersects with k_2 which also is defined in (m,n) only, the volume v_1 right above k_1 in (m,n) is actually a part of the same volume as v_3 right below k_2.

f) If an interface k_1 which is defined in (i,j) only, intersects with k_2 which also is defined in (i,j) only, the volume v_1 right above k_1 in (m,n) is actually a part of the same volume as v_3 right below k_2. The actual index of this volume must be determined by means of other parts of the interface sequences. A 'new' volume (as far as can be inferred from this two-point comparison) occurs between k_1 and k_2 in (i,j).

<u>Fig. 4</u>
Recognition of 'different volumes' as parts of a single volume.

In Fig. 4 a) the intersection curve between two interfaces is drawn. Below the intersection curve both interfaces are defined. Obviously, the space between the two interfaces and below their intersection curve constitutes a single volume.

In Fig. 4 b) the volume determination is finished for row i_1-1
The volume between the two interfaces has been named V_1 in this row. The volume sequence shall now be determined for the point $(i_1,1)$. In this point the volume between the two interfaces seems to be a 'new' one and is called V_2.

In (i_1,j_1) the volume V_1 is encountered for the first time in the volume determination for row i_1 (Fig. 4 c)).

It is recognized that V_1 and V_2 actually are parts of the same volume. In all grid points V_2 is renumbered to V_1 (Fig. 4 d)).

<u>Fig. 5</u>
The area of definition for an interface-volume pair is defined on a regular grid and consists of a set of rectangles (Fig. 5 a)). All samples on the interface where it is a boundary for the volume, must be inside the area of definition. The area of definition is also extended to include the interface boundaries.

Where two interfaces intersect, their areas of definition are made so that the non-physical extensions of the interfaces are coincident. In Fig. 5 b) this is illustrated in a vertical cross section.

Fig. 6

Example of logical volume representation build-up.

The task is to represent the volume V in Fig. 6a). This is example is 2-D, but the same procedure is used in 3D. Interface 1 is the uppermost one. SB1 is the volume below this interface (Fig. 6b)). As this interface does not intersect with any other interface which is above the volume, no simple blocks shall be subtracted from SB1 at this stage. Thus in the first step:

V = SB1

The next interfaces are number 2 and 3. Their physical parts are not overlapping, so both interfaces are used in this step. The simple blocks SB2 and SB3 are shown in Fig. 6c). Interface 2 and 3 are both locally below the volume and the non-physical interface extensions are above the interfaces. Hence two extra SBs must be made: SBH1 = SB(+2,-3) and SBH2 = SB(-2,+3). They are subtracted from the original SB2 and SB3:

SB2 = SB2 - (SBH1 U SBH2)
SB3 = SB3 - (SBH1 U SBH2)

SB2 and SB3 are now subtracted from V:

V = V - (SB2 U SB3)

SBH1, SBH2 and SB2 U SB3 are shown in Fig. 6d. 'Current volume' V is shown in Fig. 6e.

Next interface is number 4. SB4 = SB(+4) is shown in Fig. 6f). This SB is added to the volume:

V = V U SB4

Fig. 6g) shows 'current volume' V.

Finally SB5 = SB(+5) (Fig. 6h)) is subtracted:

V = V - SB5

and the volume of Fig. 6a) is obtained.

Fig. 7

Removal of small 'errors' in the set of interfaces. The removal is illustrated in vertical cross sections.

Fig. 7a):

The samples of two interfaces are marked with x and o. The hatched lines indicate the linear inter- and extrapolations of interest. # is their intersection point. This is regarded as the interface intersection point in this particular cross section. A 'missing' sample will be added at the arrow.

Fig. 7b):

This figure is similar to 7a), but in this case a sample is removed at the arrow.

Fig. 8

Example of volume distorted by B-spline oscillation.

The volume of interest in the one between interface 1 and 2 in Fig.8a). The B-spline representations are shown in Fig. 8b). The interfaces are drawn inside their areas of definition. The volume (hatched area) is erroneous with this representation because the interfaces do not intersect.

In Fig. 8c) samples are added by linear extrapolation. Now there is complete 'data control' within the entire area of definition. The B-splines made from this new set of samples are shown in Fig. 8d). The volume is now correct.

Fig. 9

Example of volume logic build-up with global areas of definition.

Fig. 9a) shows the volume to be made (hatched area) and the interfaces and their global areas of definition.

The volume V1 (Fig. 9b)) is made with the procedure described in Section 2.4, but global areas of definition have been used.

The same procedure used from bottom and upward produces the volume V2 shown in Fig. 9c).
Fig. 9d) shows the volume V3 defined by the volume area of definition.

The final volume V is the intersection of the three volumes:

$$V = V1 \cap V2 \cap V3$$

Fig. 10

Example of volume that can not be represented by means of the volume logic described in section 3.4 and Figure 9. With this volume complexity, a separate area of definition must be made for each interface-volume pair. The volume can then be represented by means of the logic described in section 2.4.

The hatched area in Fig. 10a) indicates the volume that shall be represented. The procedure of section 3.4 produces the volume shown in Fig. 10 b).

Fig. 1 a)

Invalid interface set

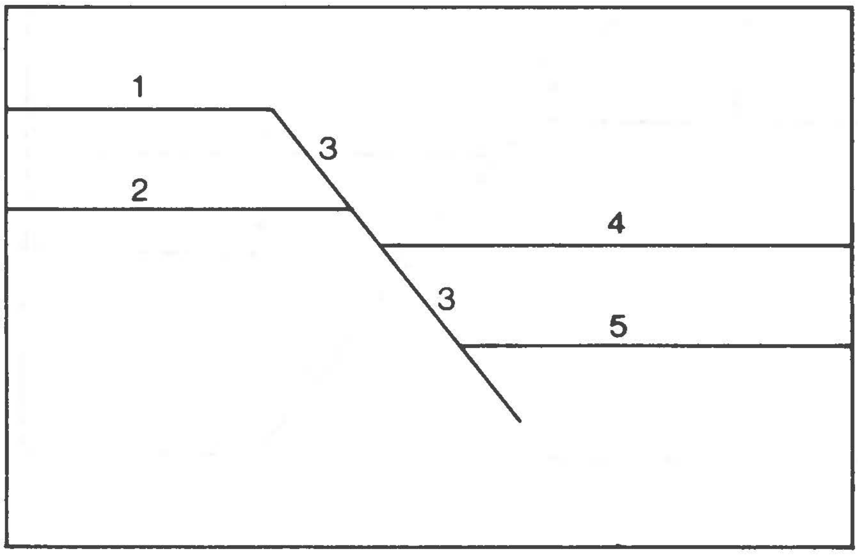


Fig. 1 a) (forts.)

Valid interface set

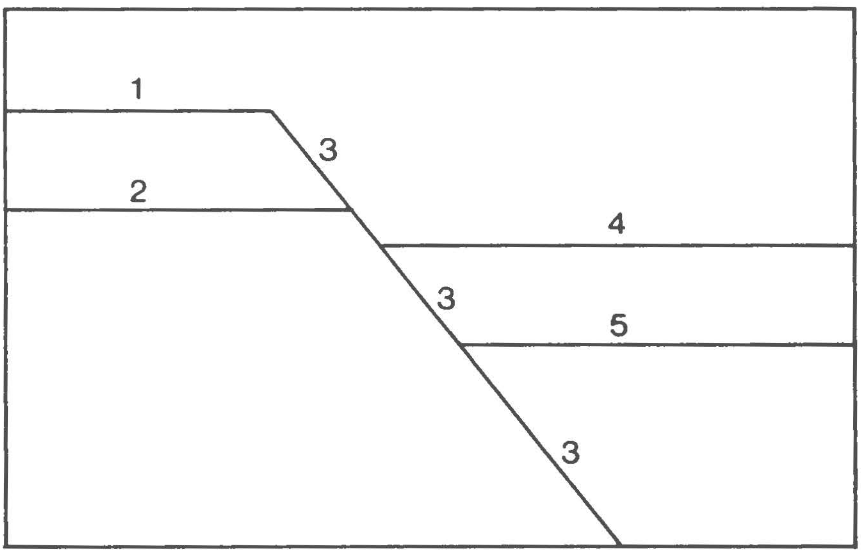


Fig. 1 a) (forts. 2)

Valid interface set

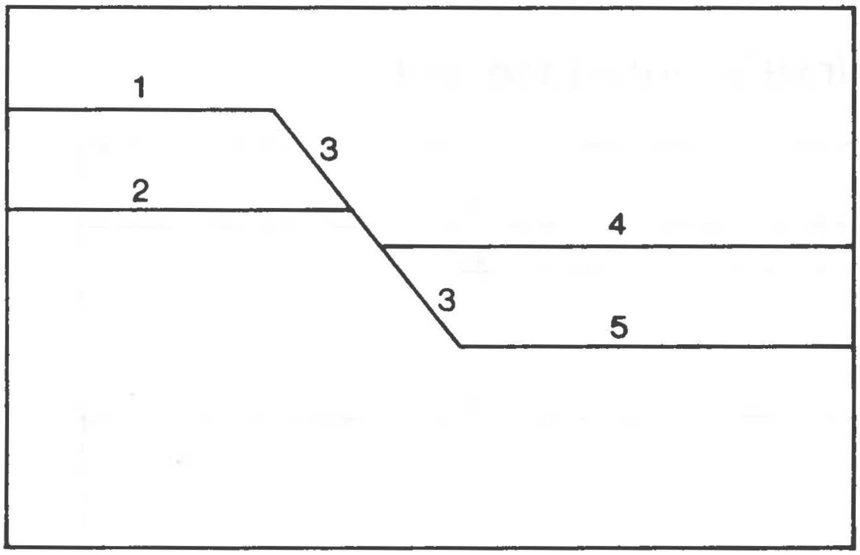


Fig 1 b)

Invalid interface set

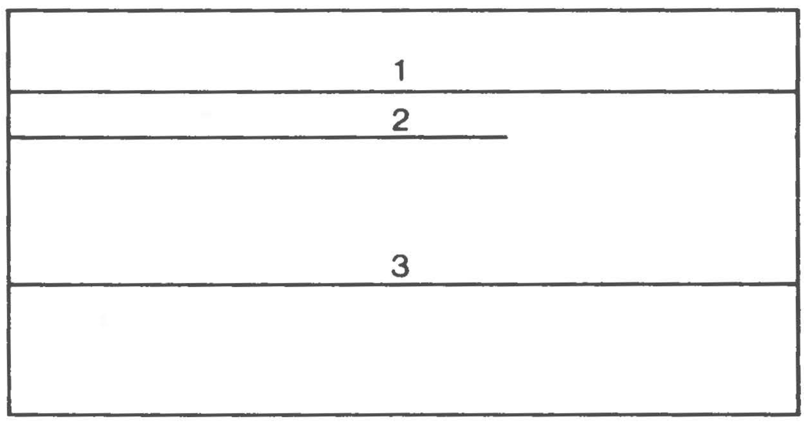


Fig 1 b) (forts.)

Valid interface set

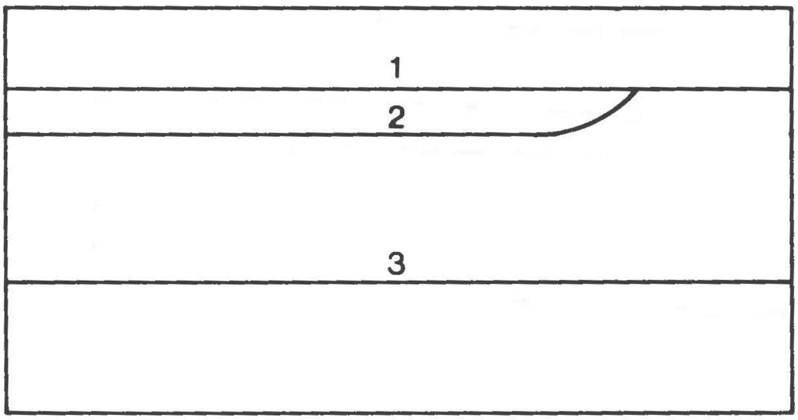


Fig 1 b) (forts. 2)

Valid interface set

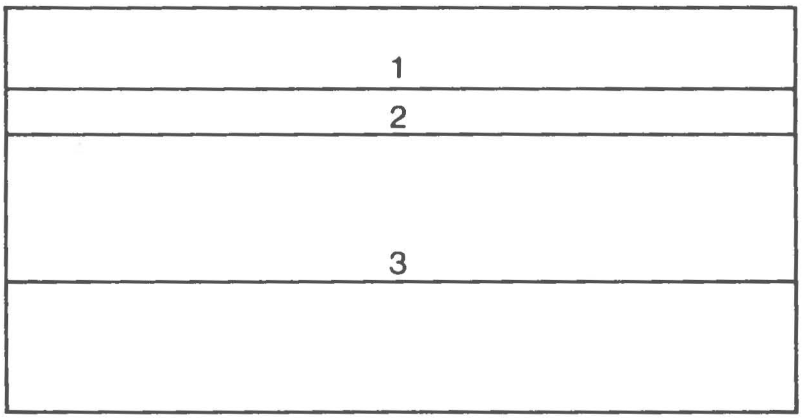


Fig. 1 c)

Invalid interface set.

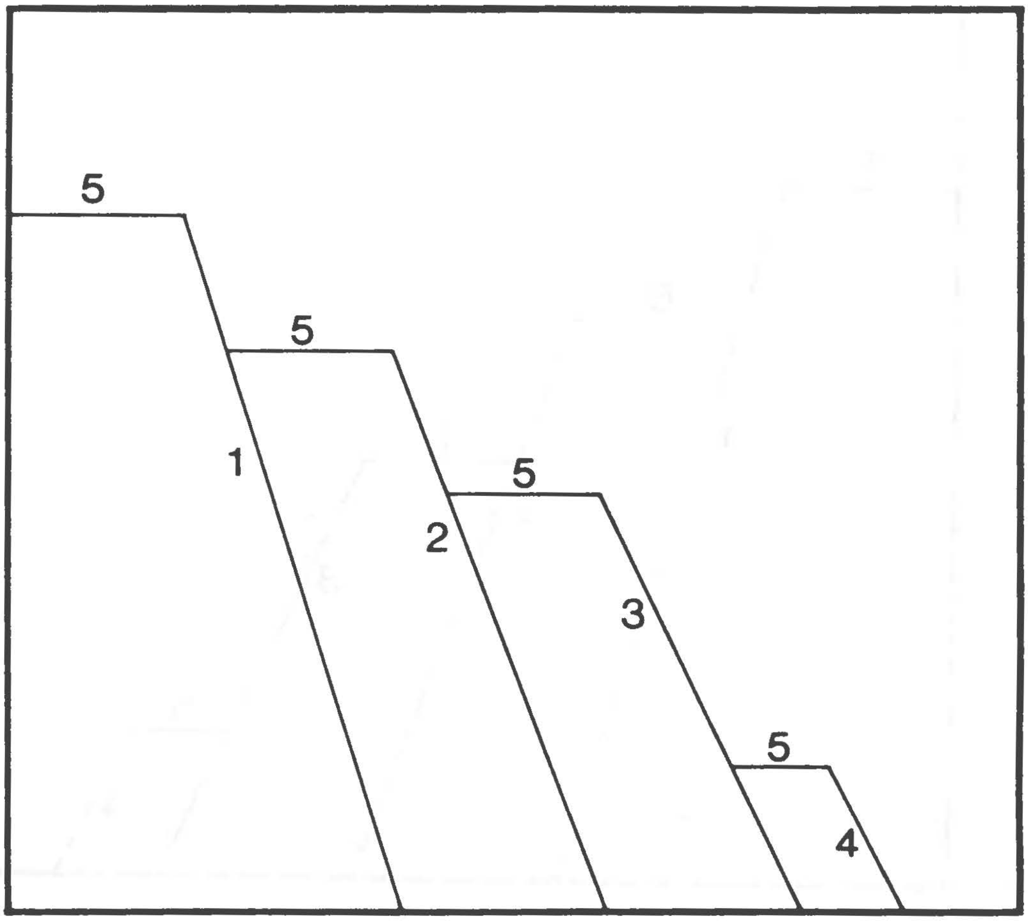


Fig. 1 c) (forts.)

Valid interface set.

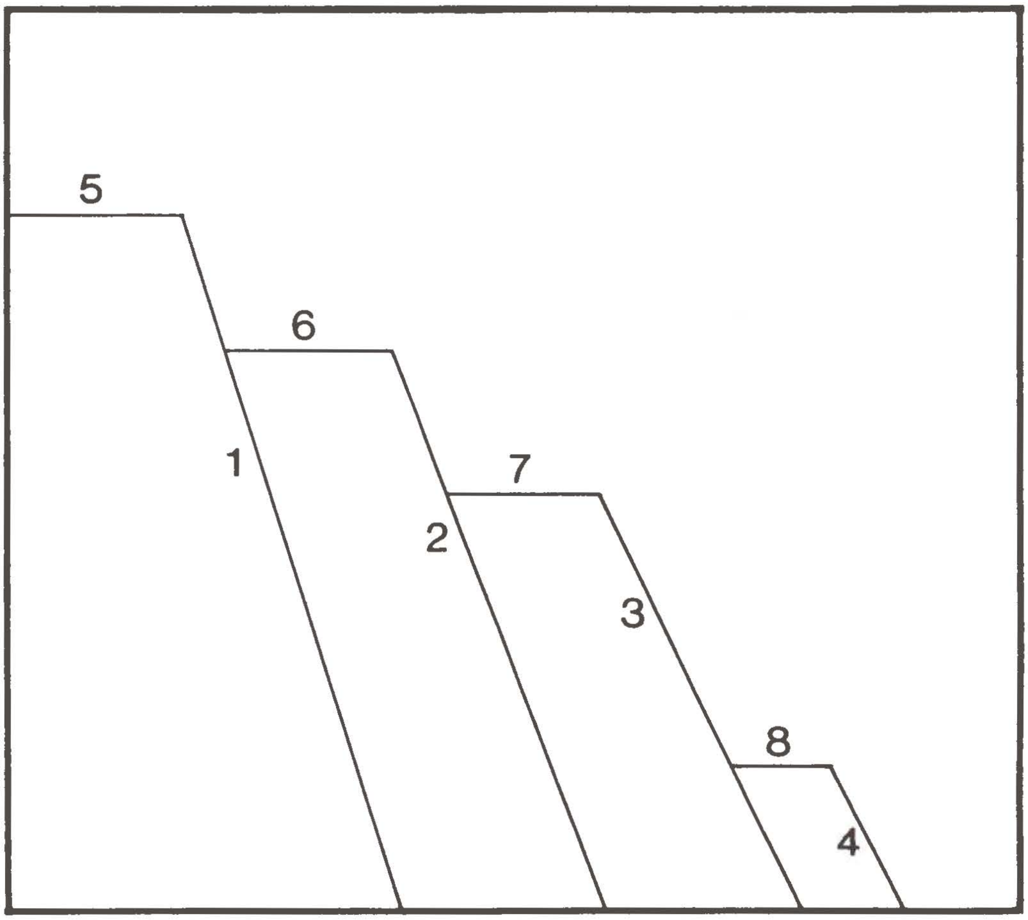


Fig. 2

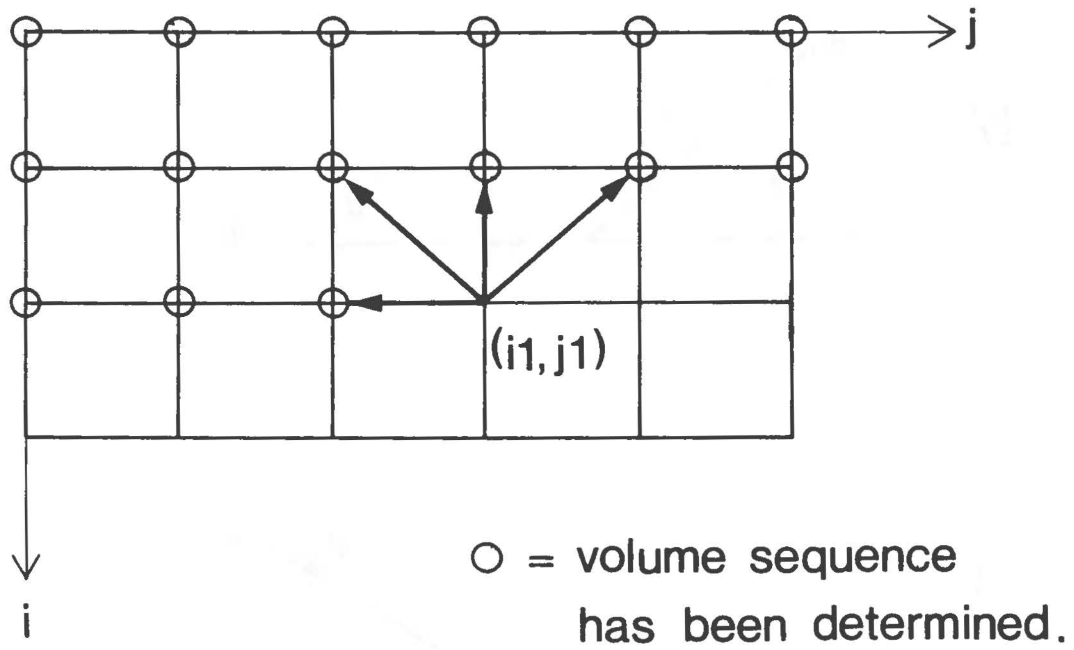
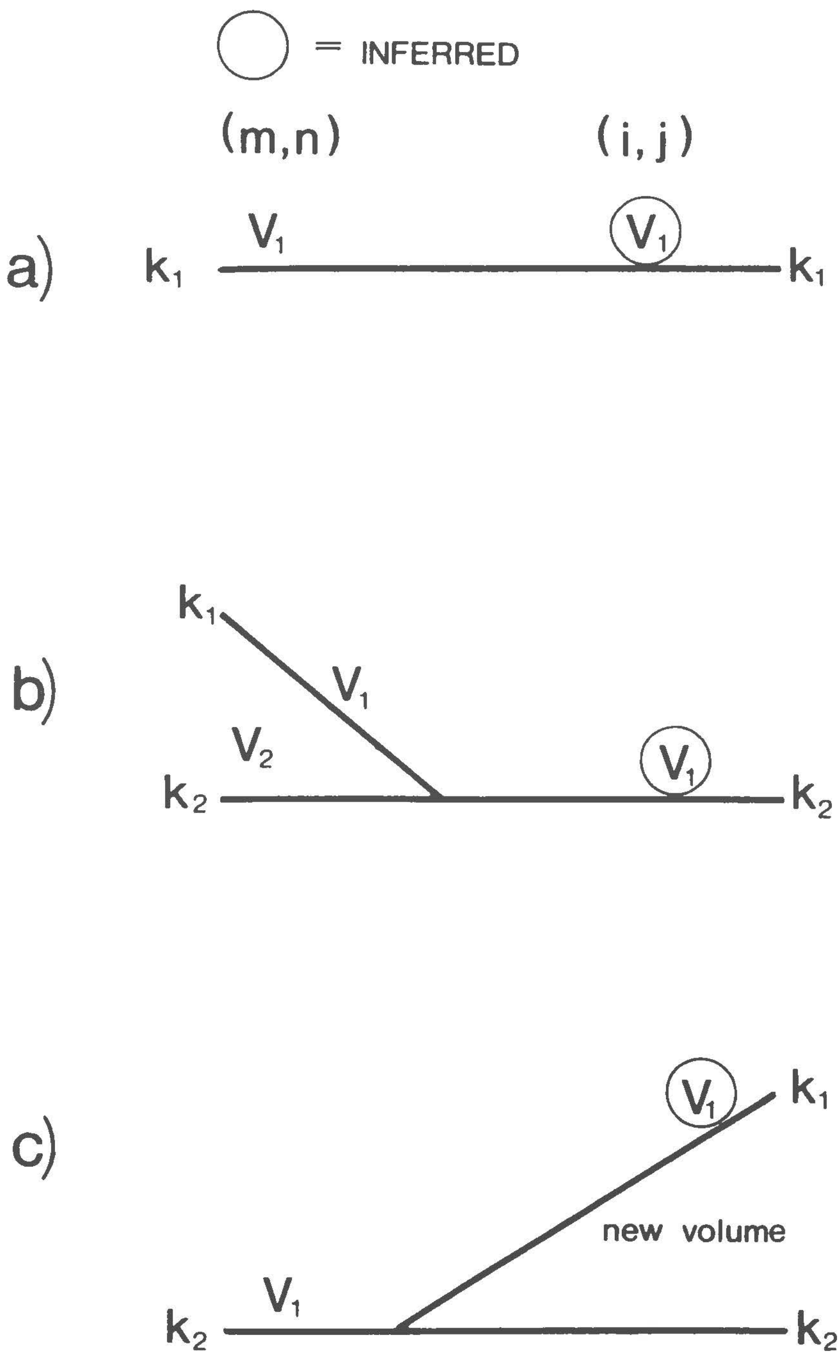


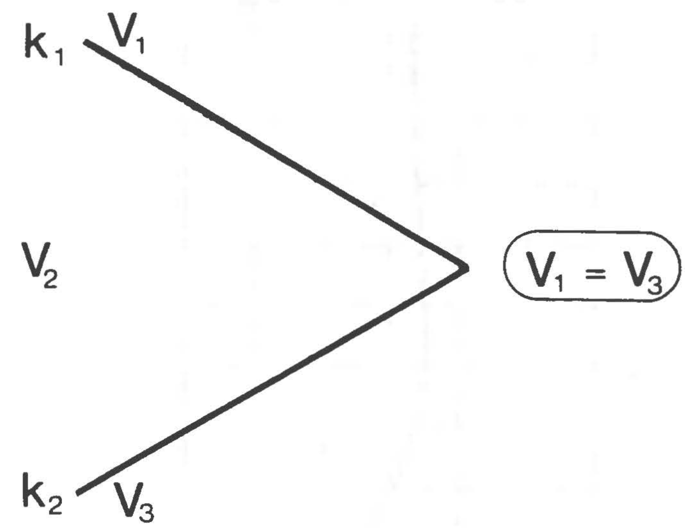
Fig. 3


= INFERRED
(m,n)
(i,j)
a)
k_1
V_1
V_1
k_1
b)
k_1
V_1
V_2
k_2
V_1
k_2
c)
V_1
k_1
new volume
k_2
V_1
k_2

(m, n) (i, j)

d)

e)



f)

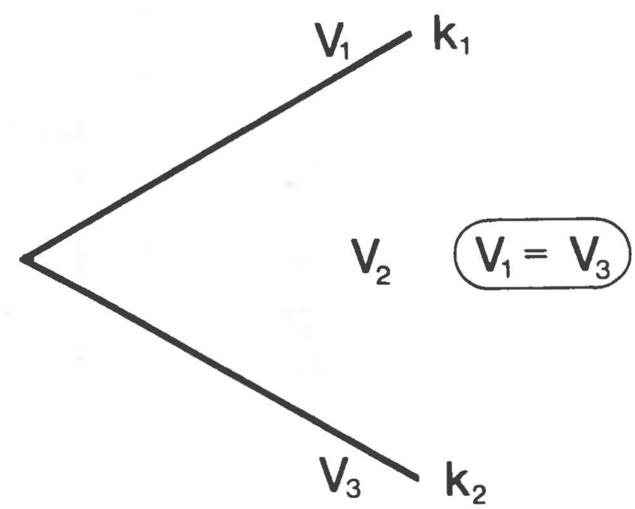


Fig. 4 a)

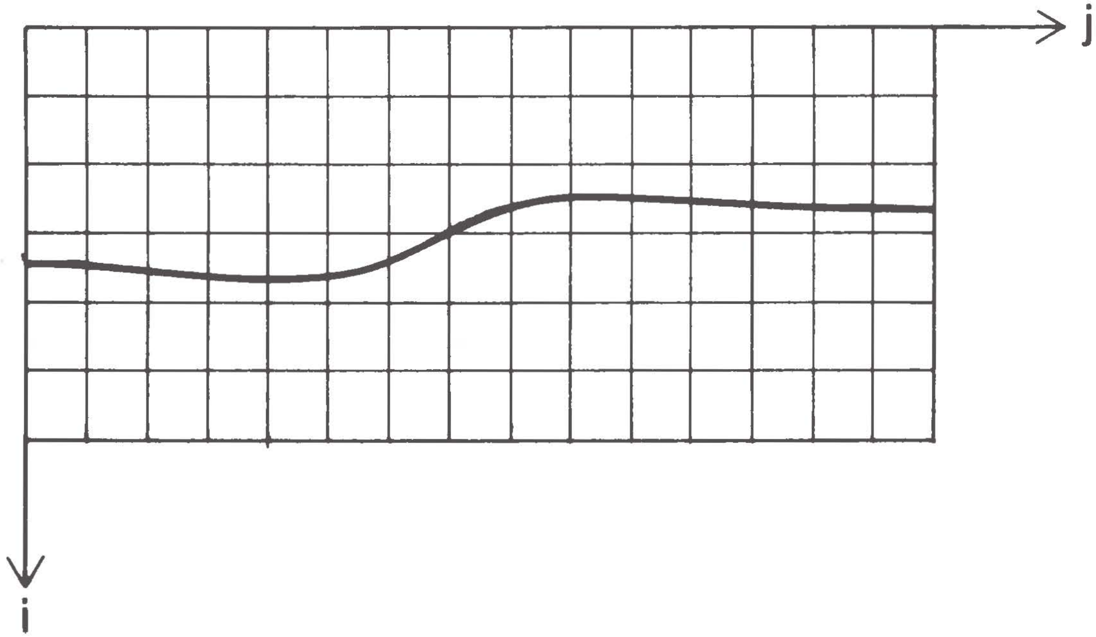


Fig. 4 b)

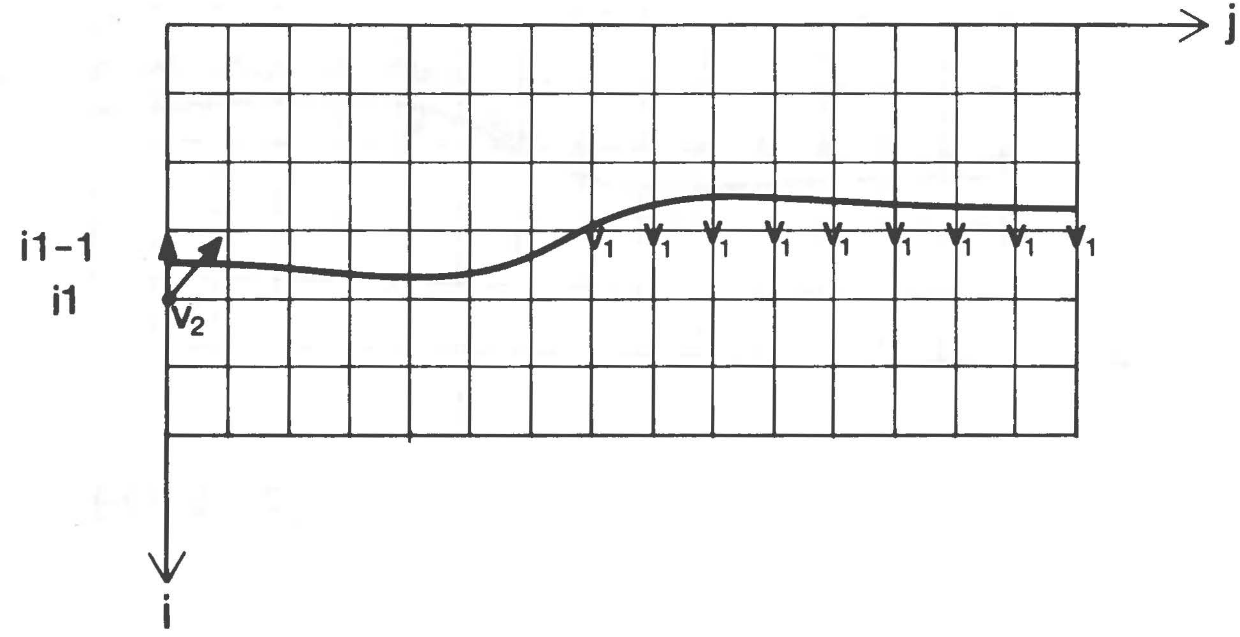


Fig. 4 c)

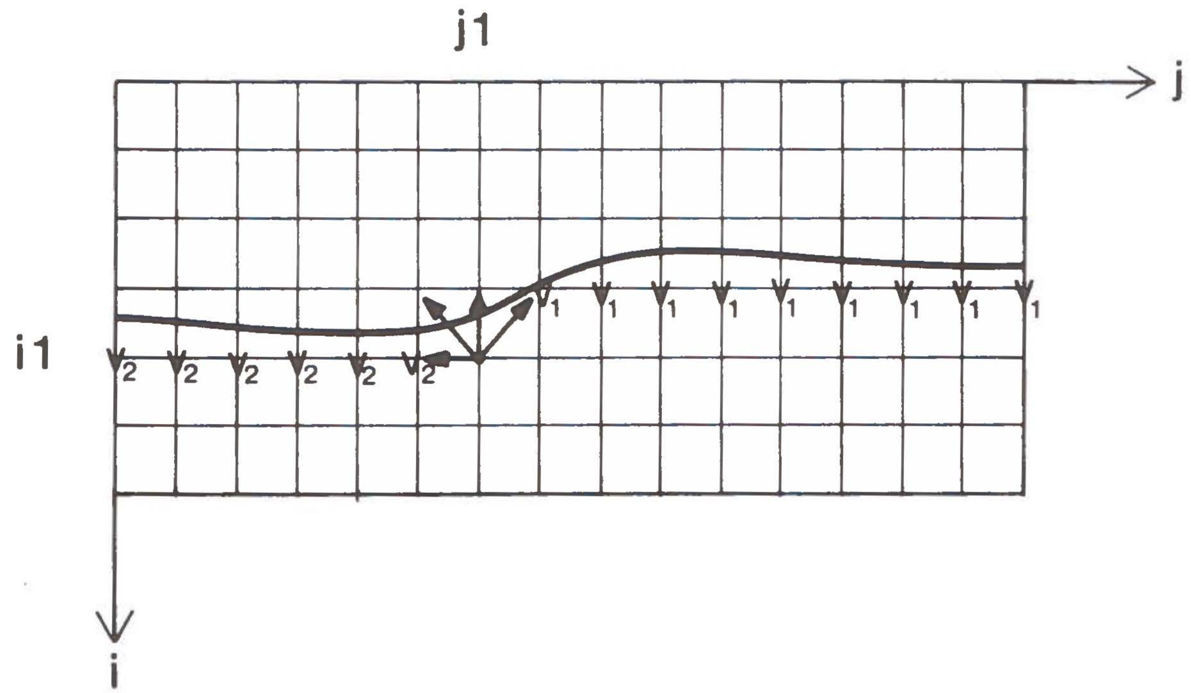


Fig.4 d)

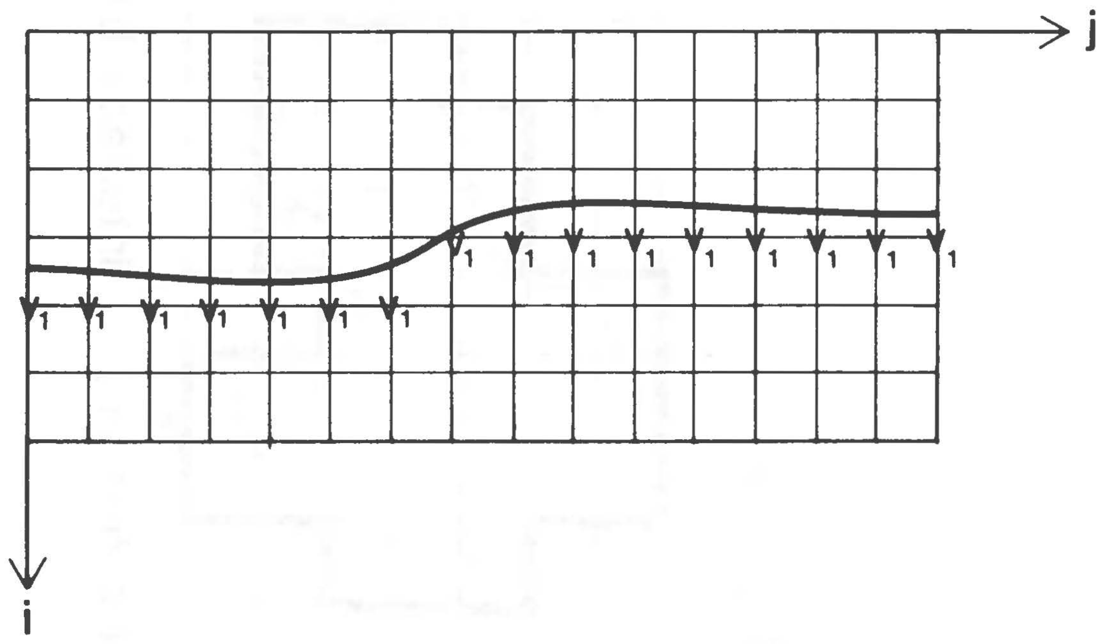


Fig. 5 a)

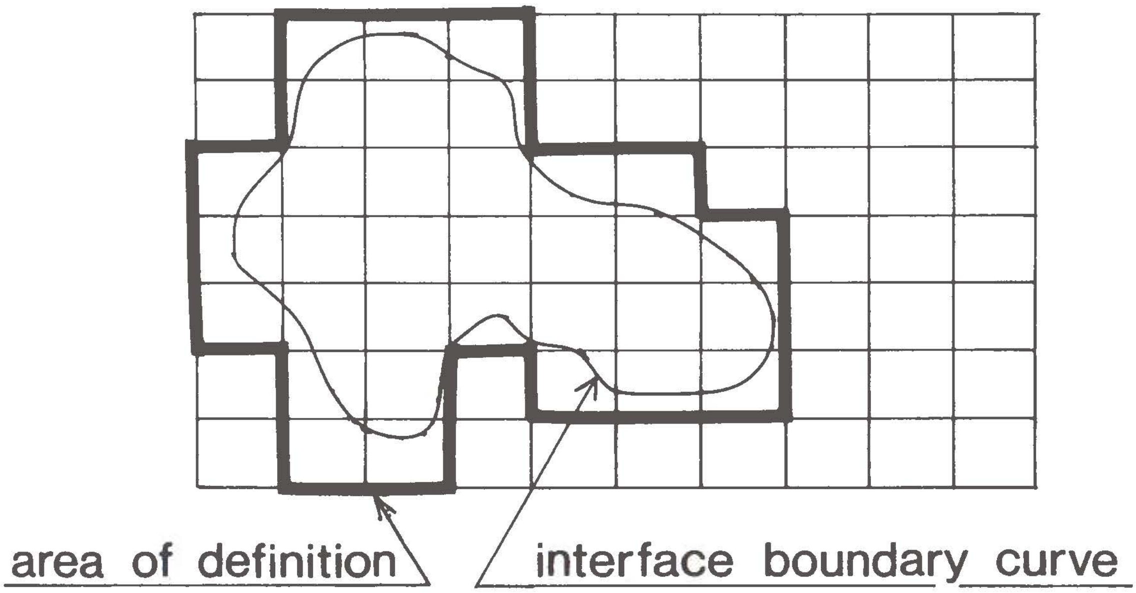


Fig. 5 b

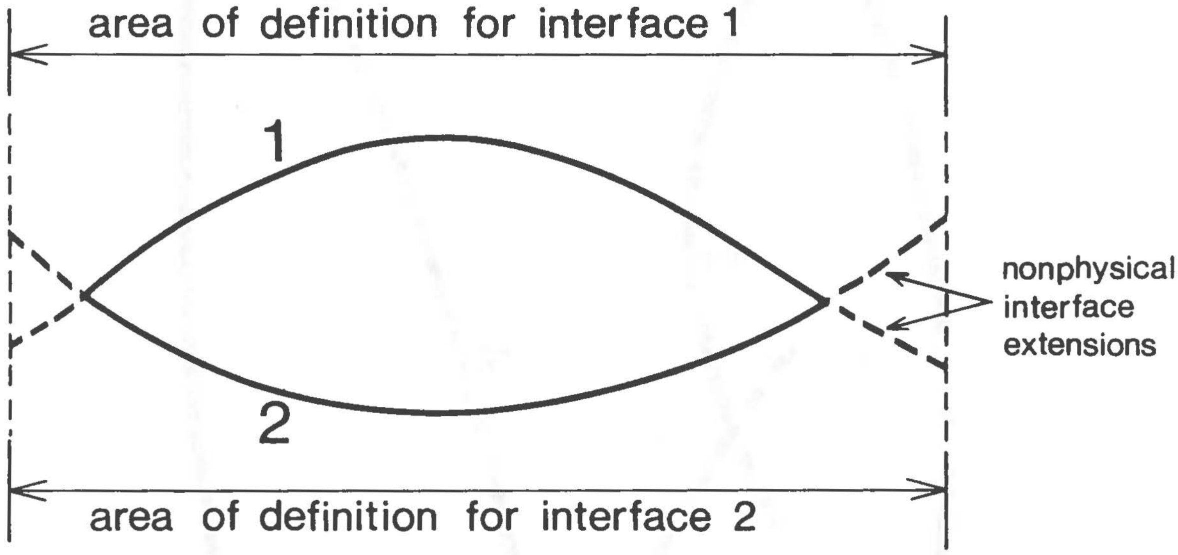

area of definition for interface 1
1
2
nonphysical
interface
extensions
area of definition for interface 2

Fig. 6

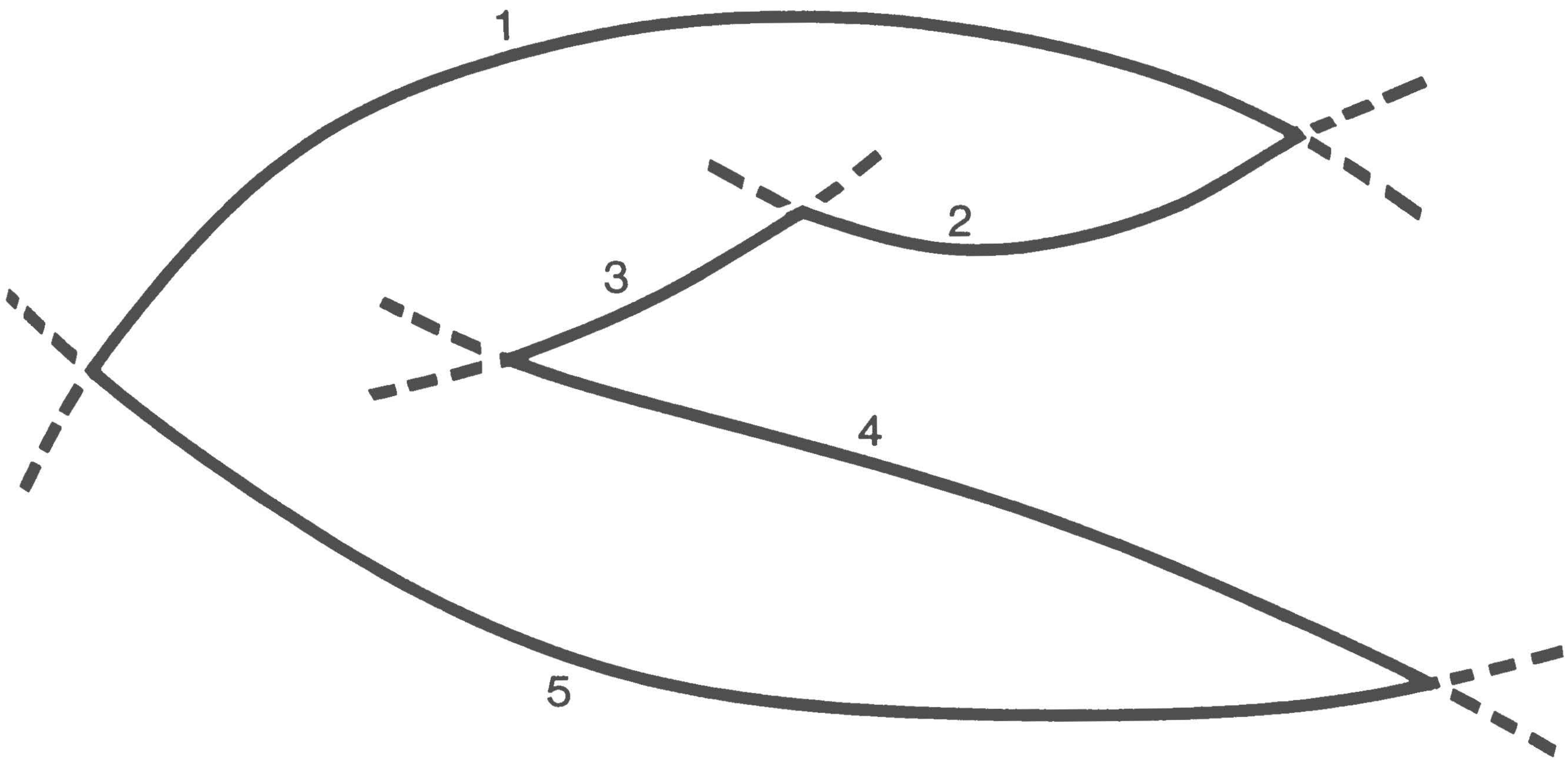


Fig. 6 a)

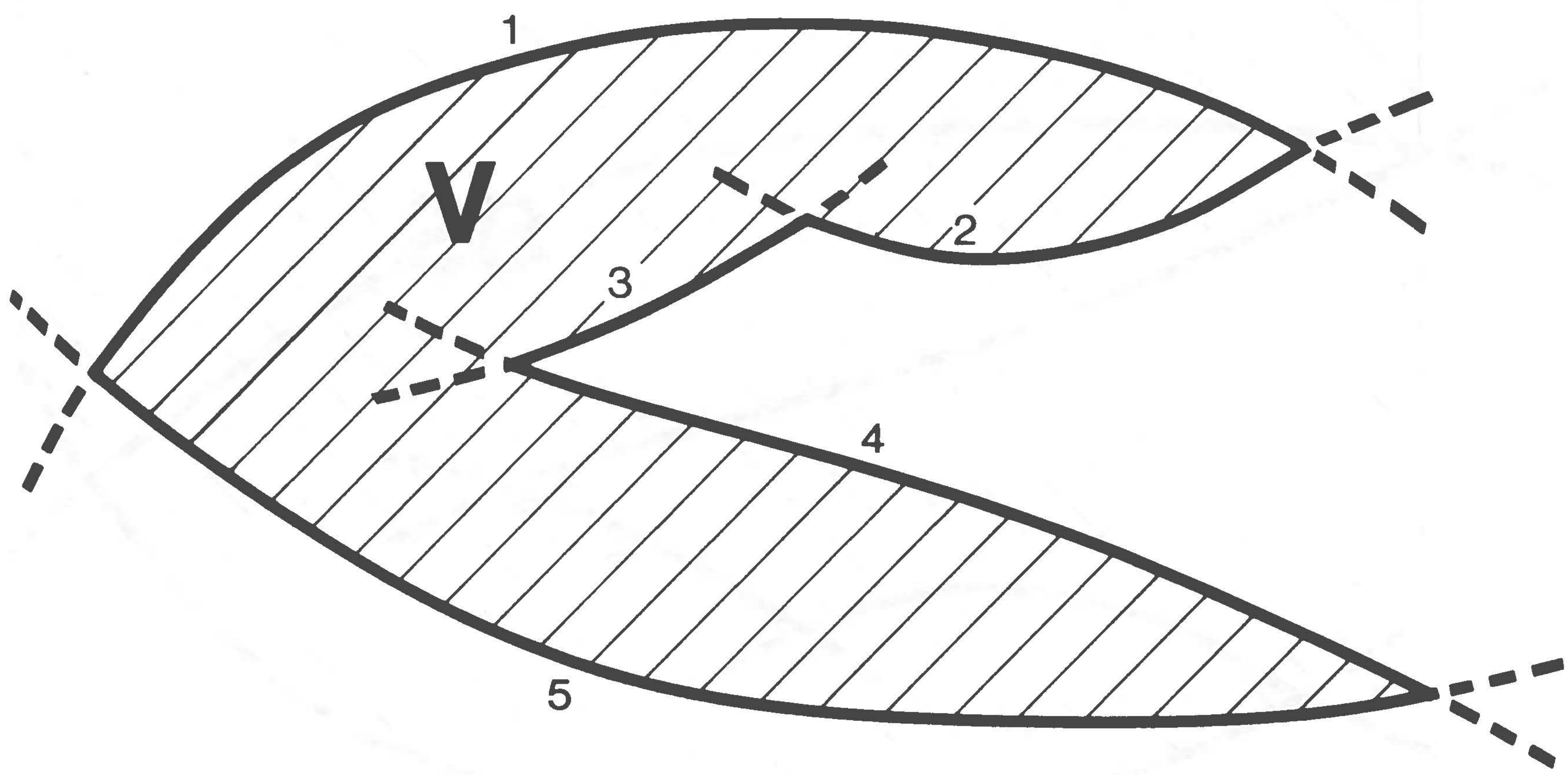


Fig. 6 b)

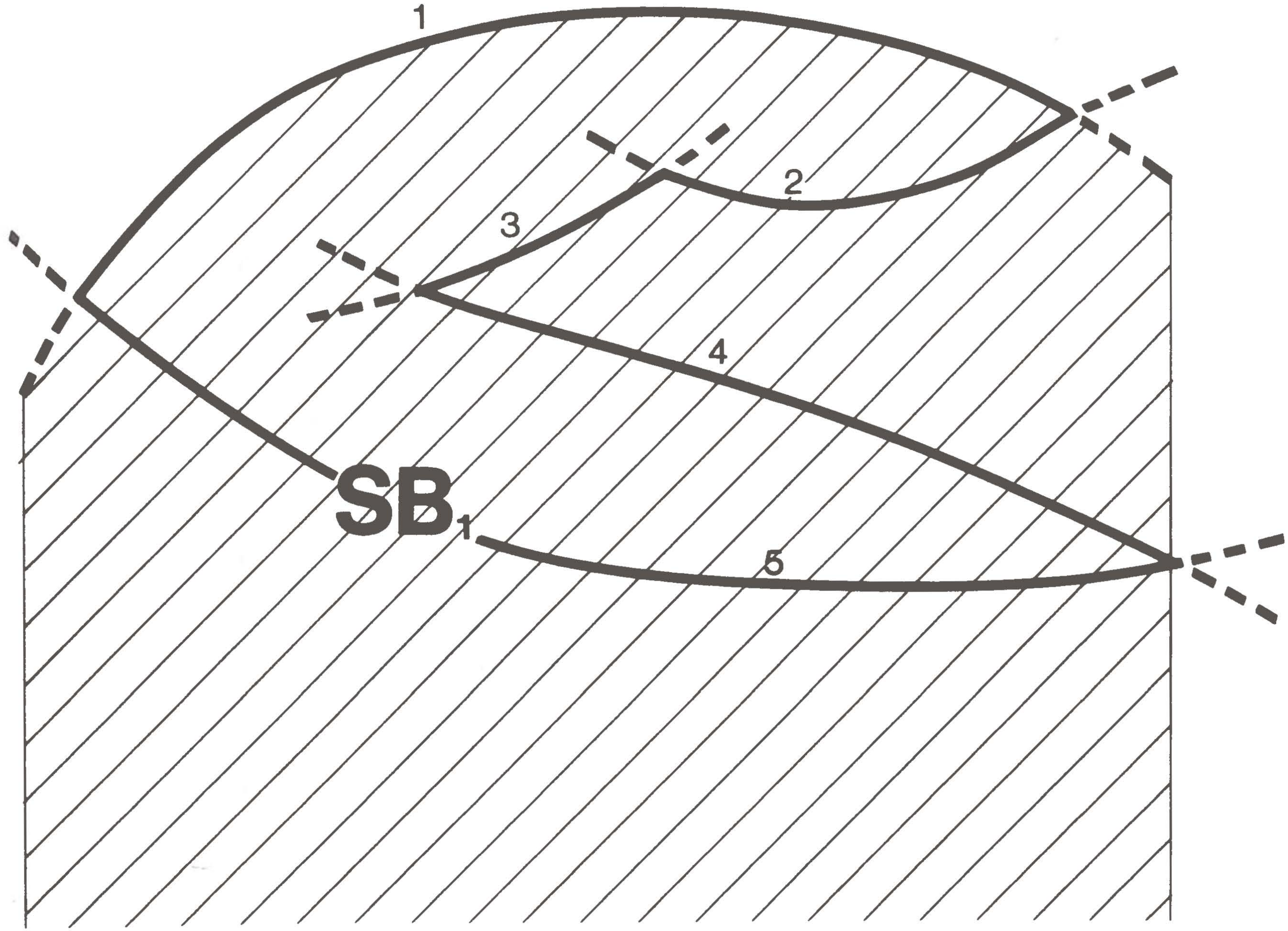


Fig. 6 c)

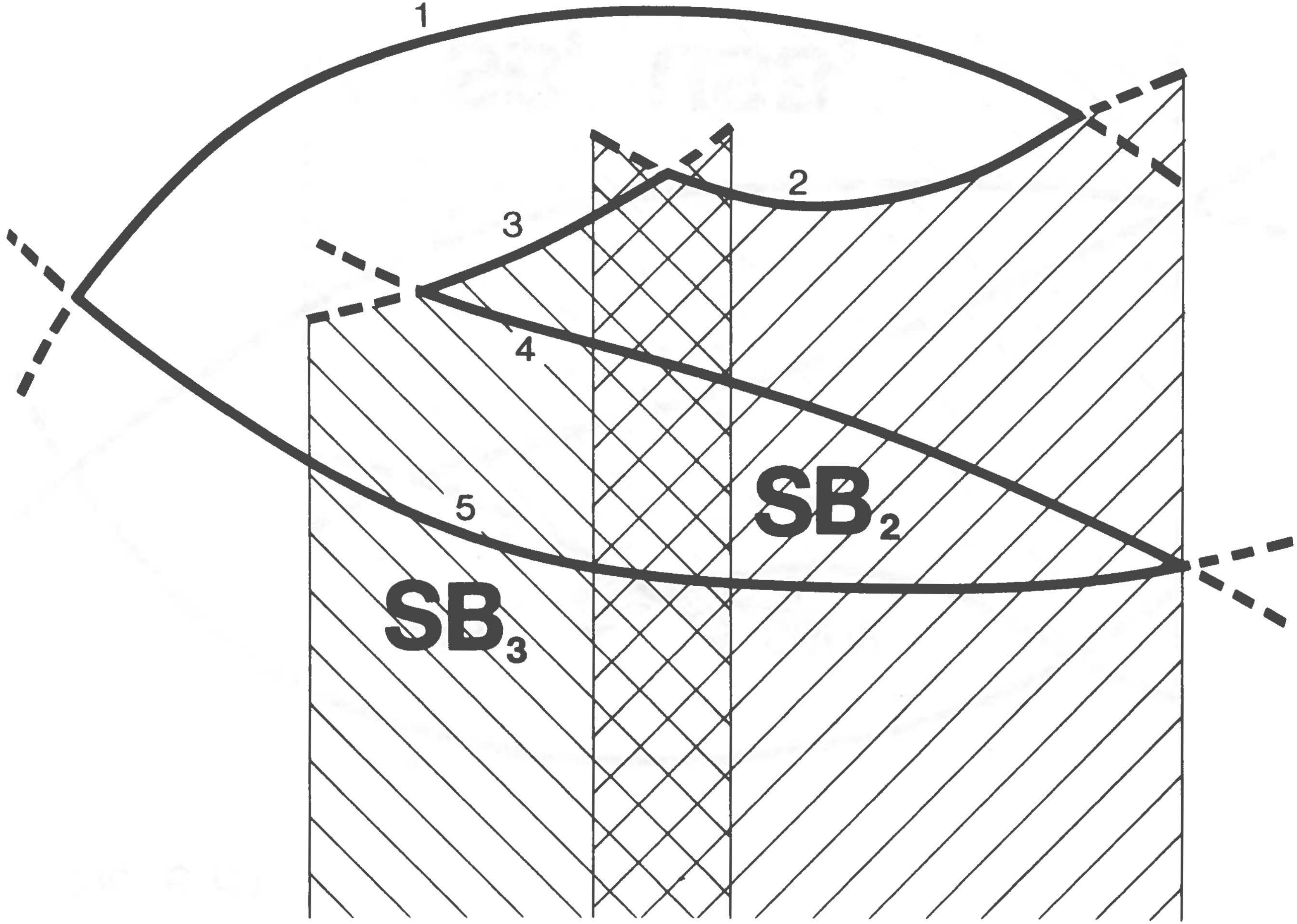


Fig. 6 d)

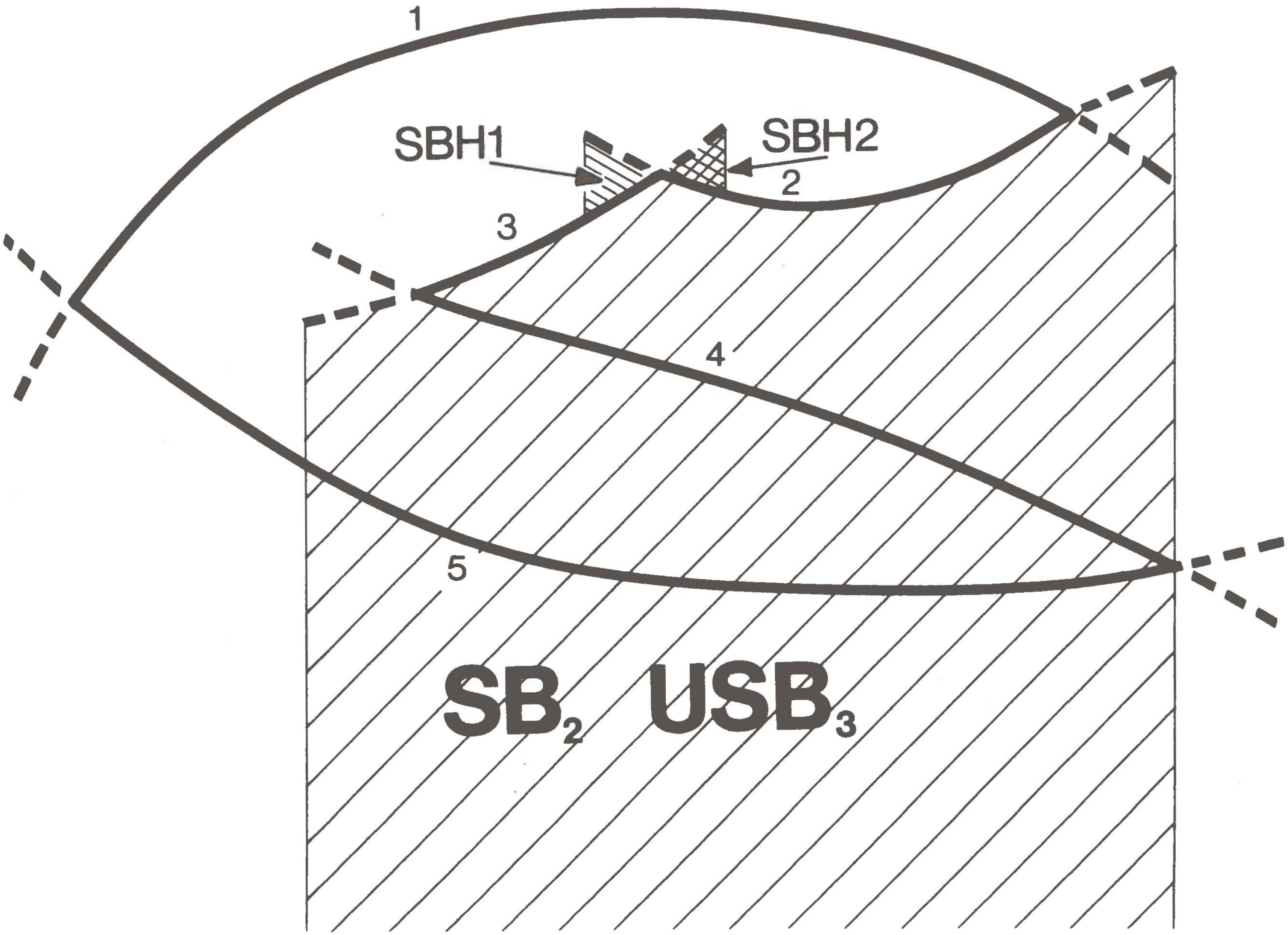


Fig. 6 e)

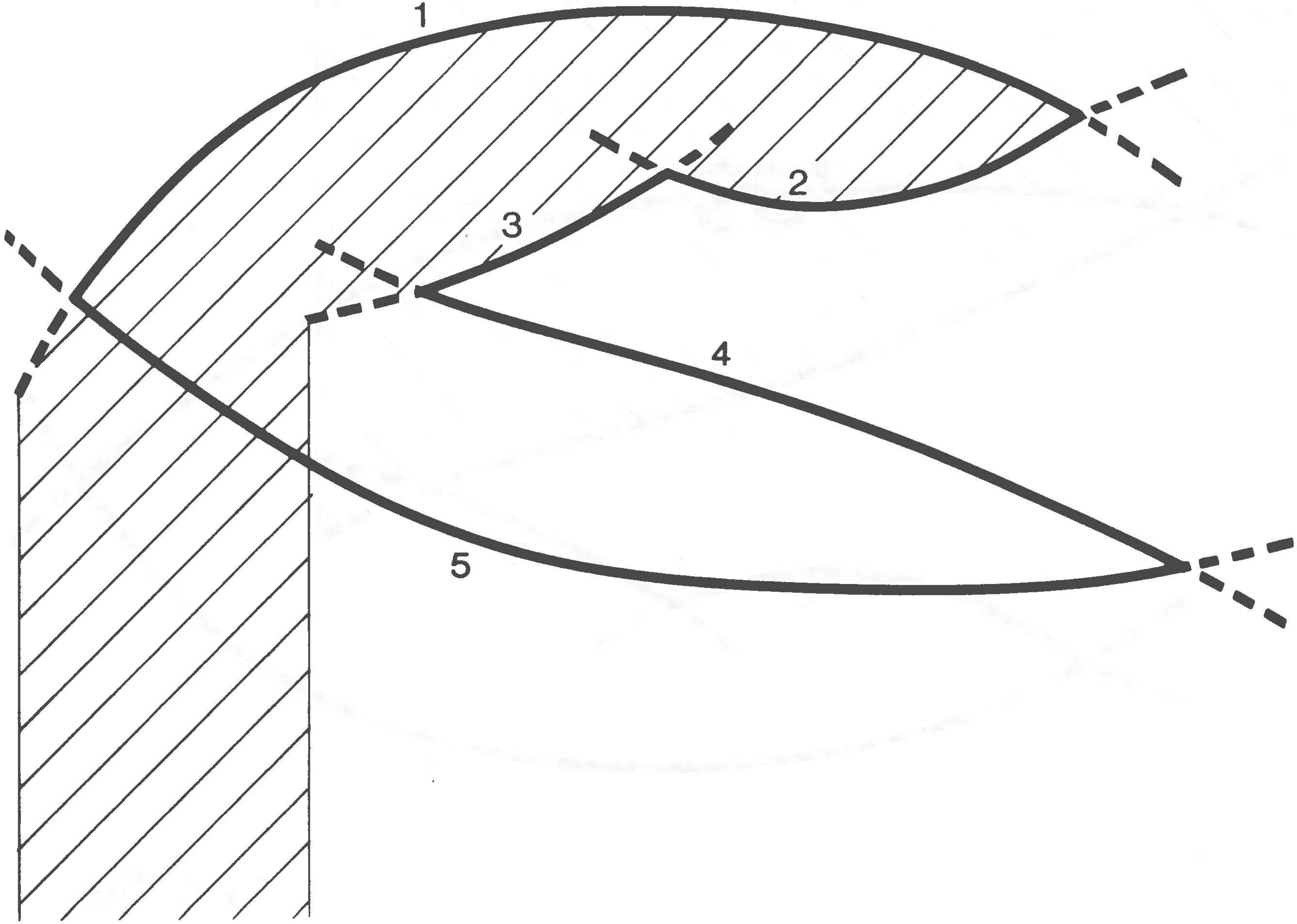


Fig. 6 f)

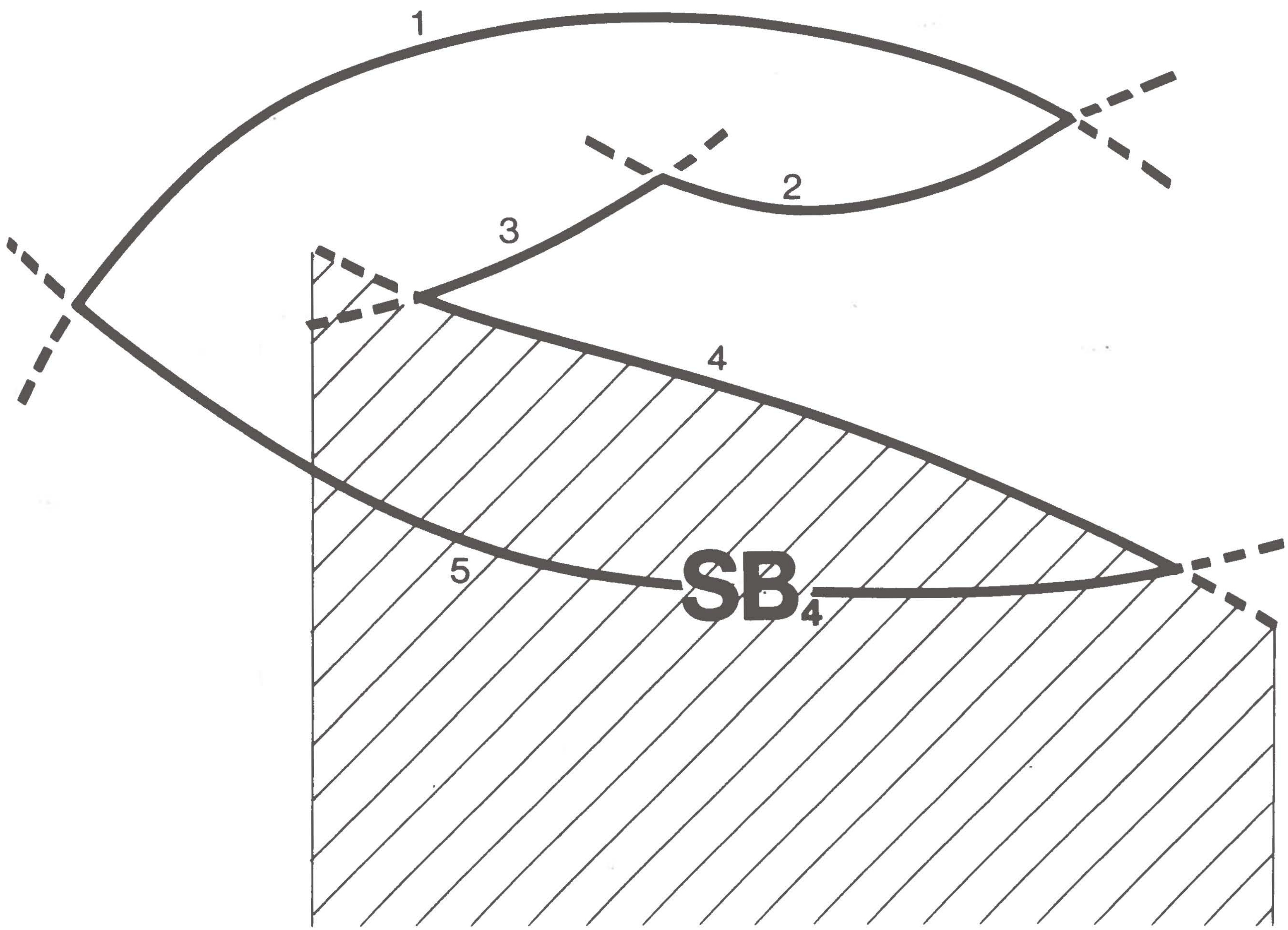


Fig. 6 g)

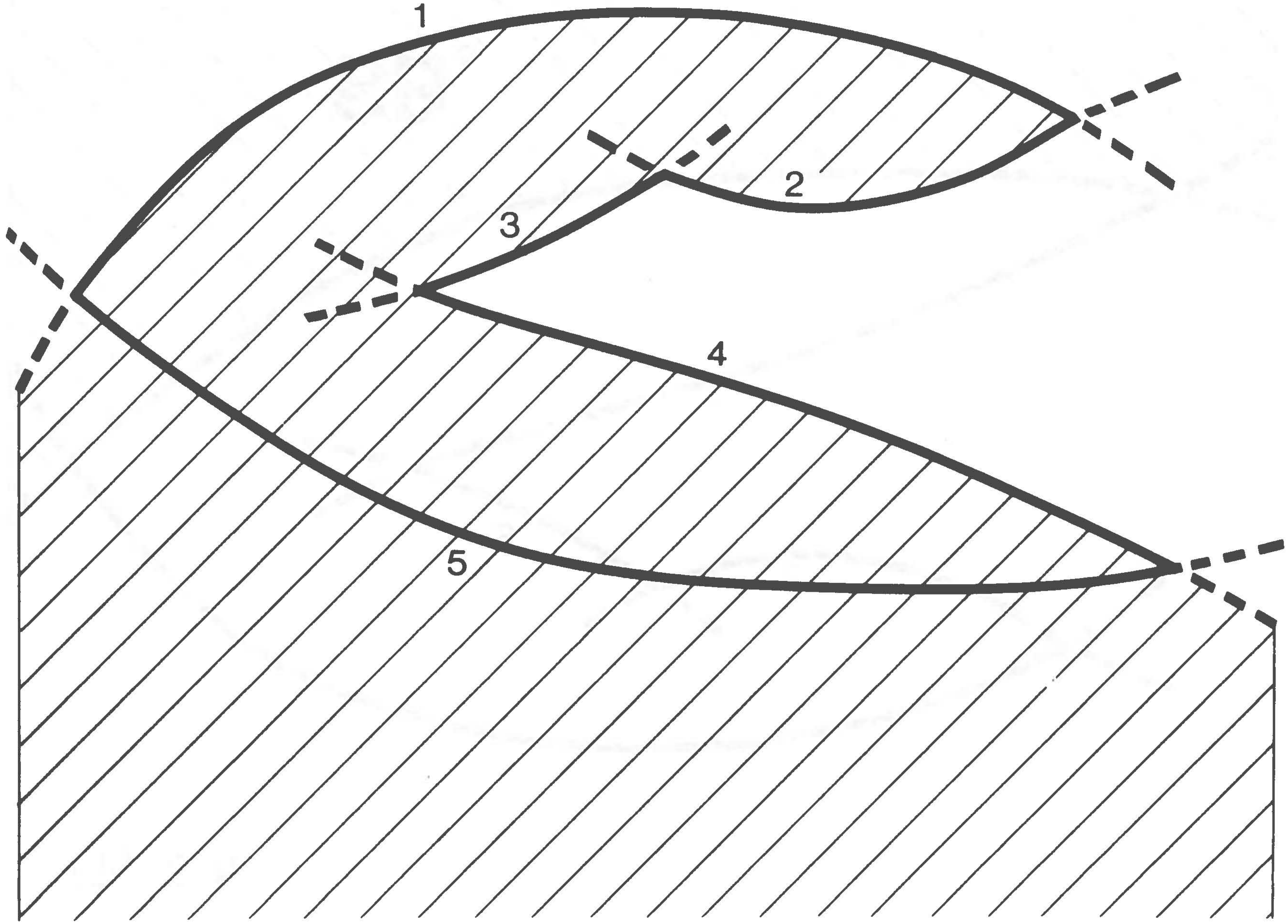


Fig. 6 h)

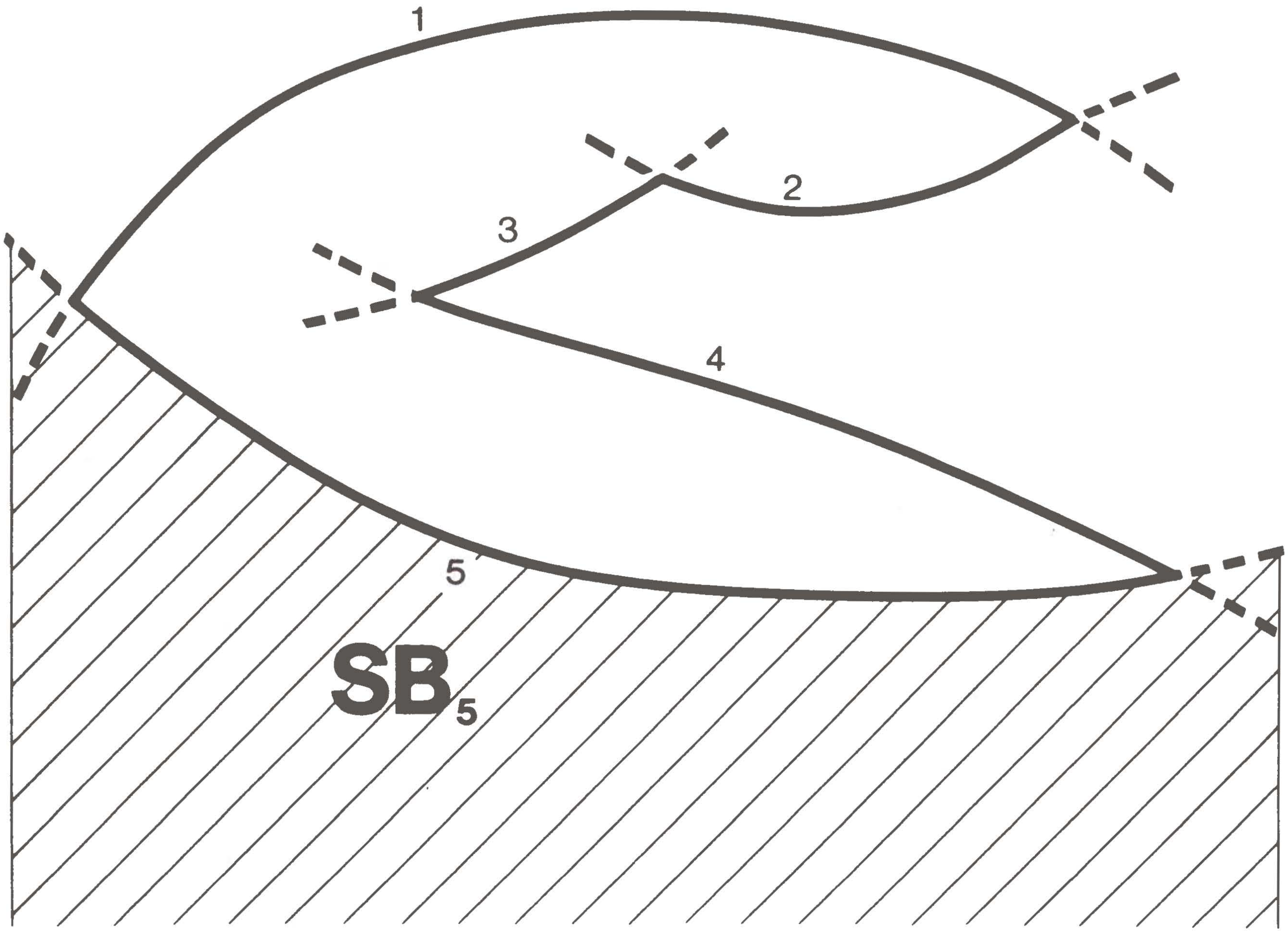
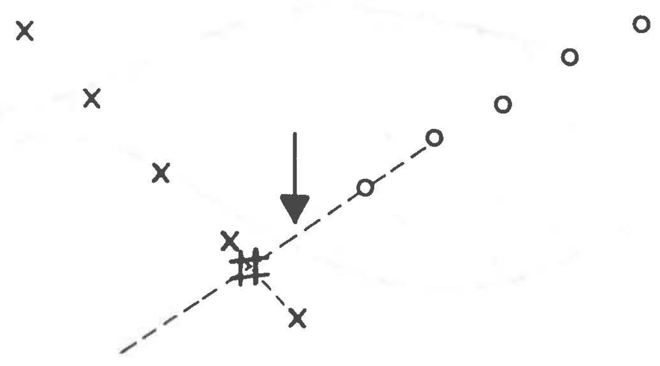


Fig. 7 a)



7 b)

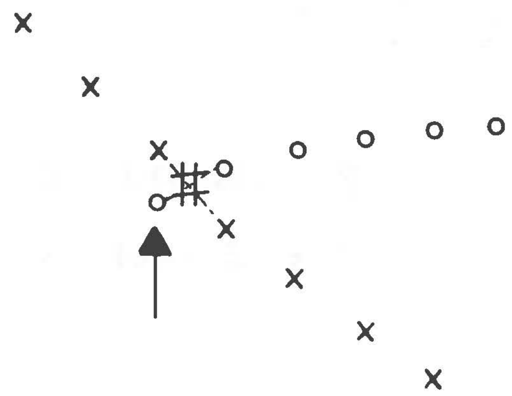
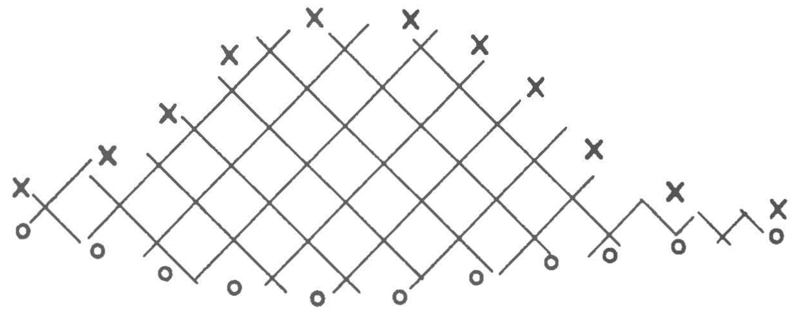


Fig. 8

x = interface 1

o = interface 2

a)



b)

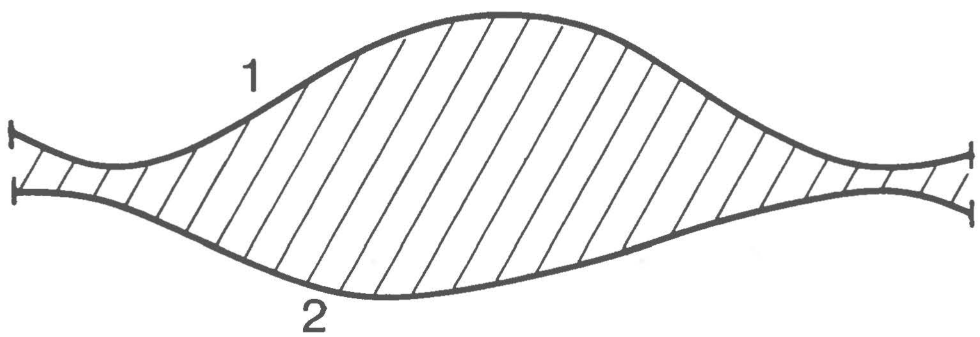
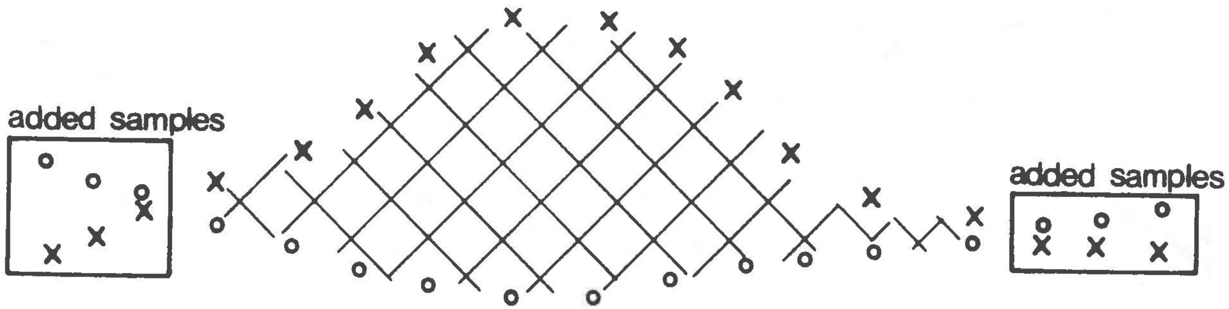


Fig. 8

x = interface 1

o = interface 2

c)



d)

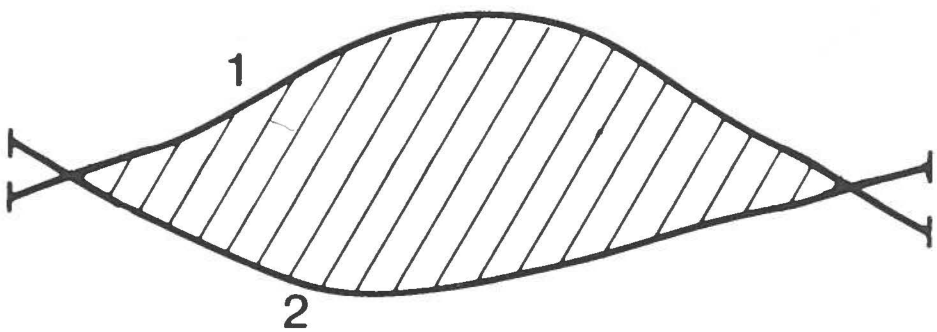


Fig. 9 a)

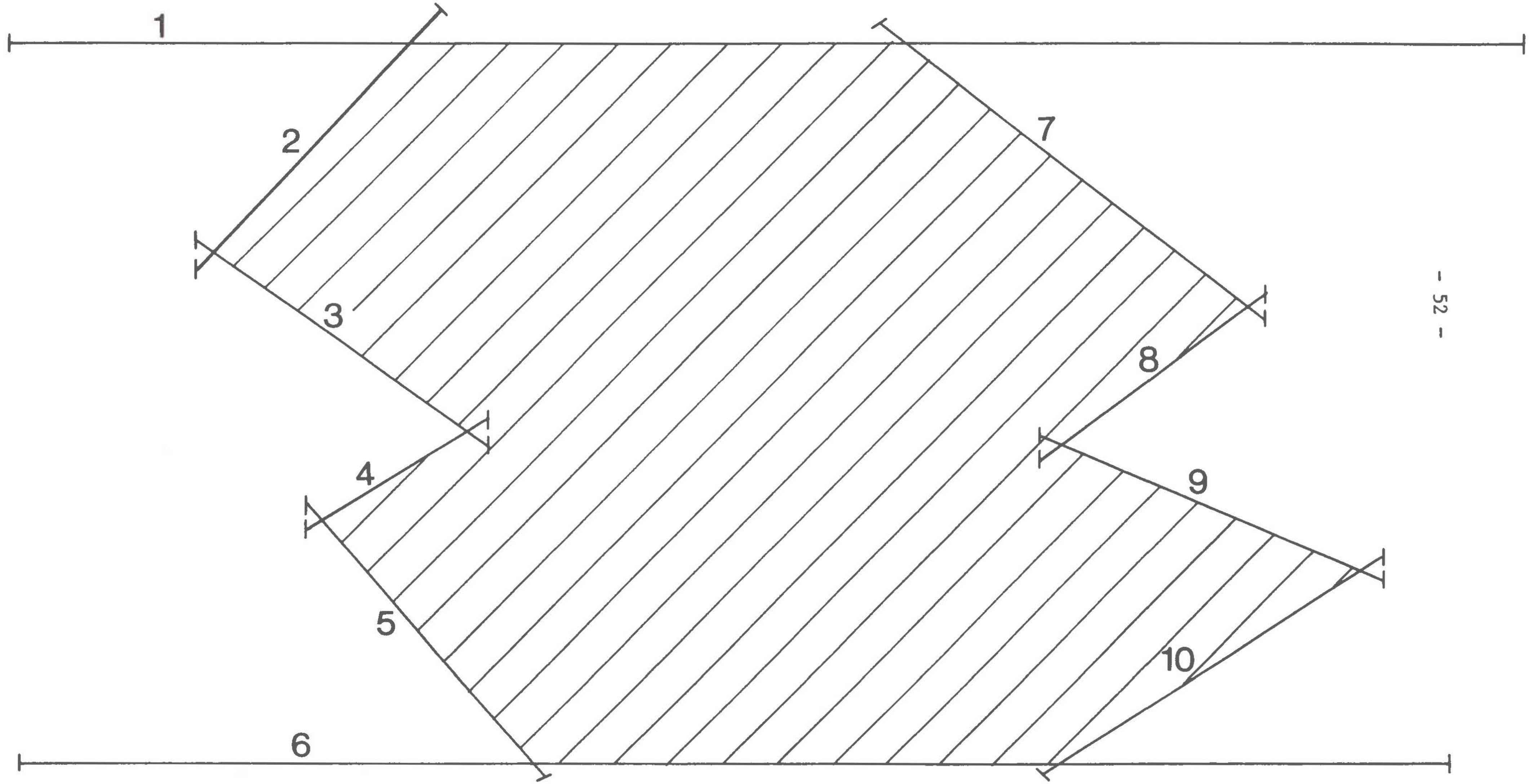


Fig. 9 b)

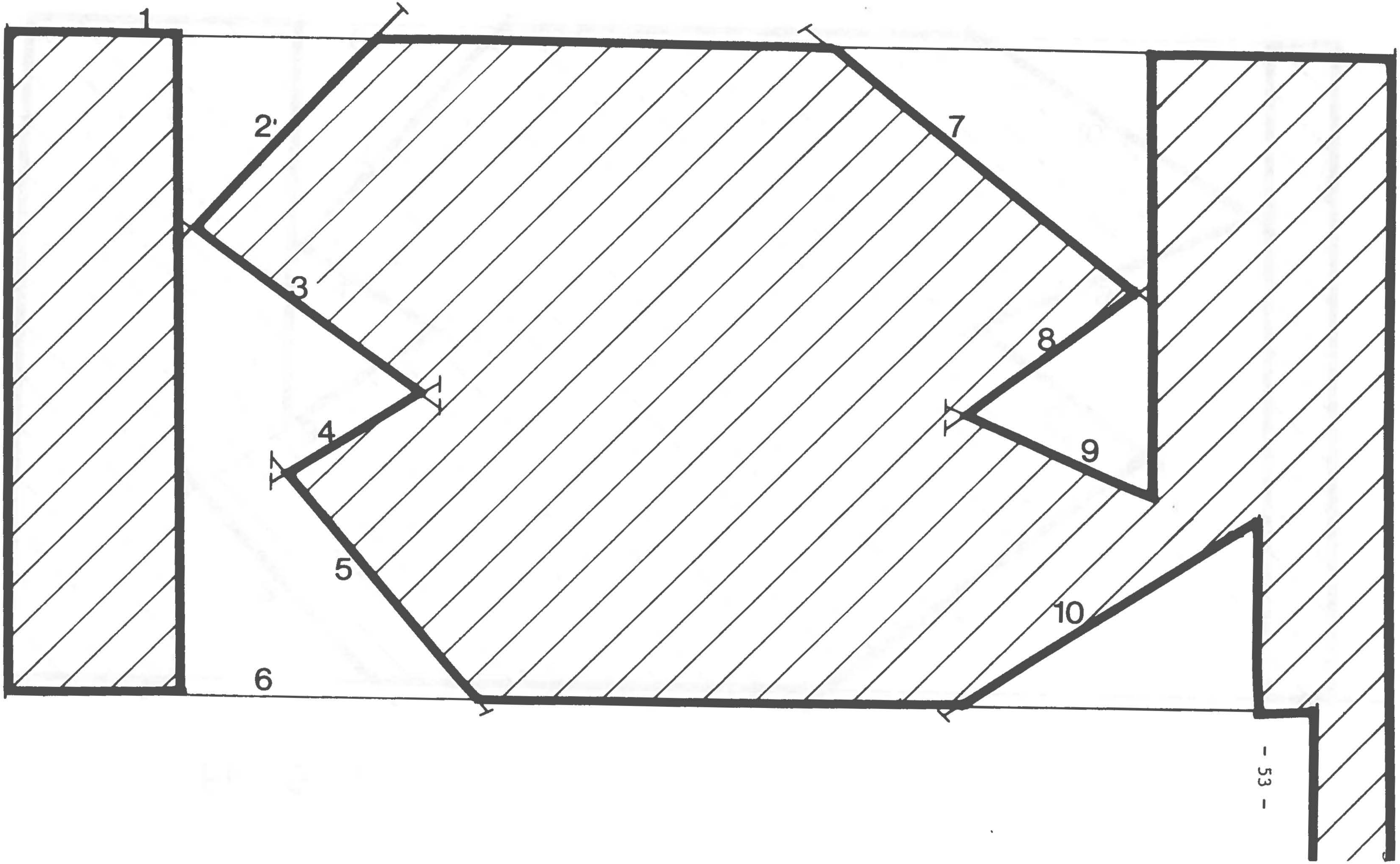
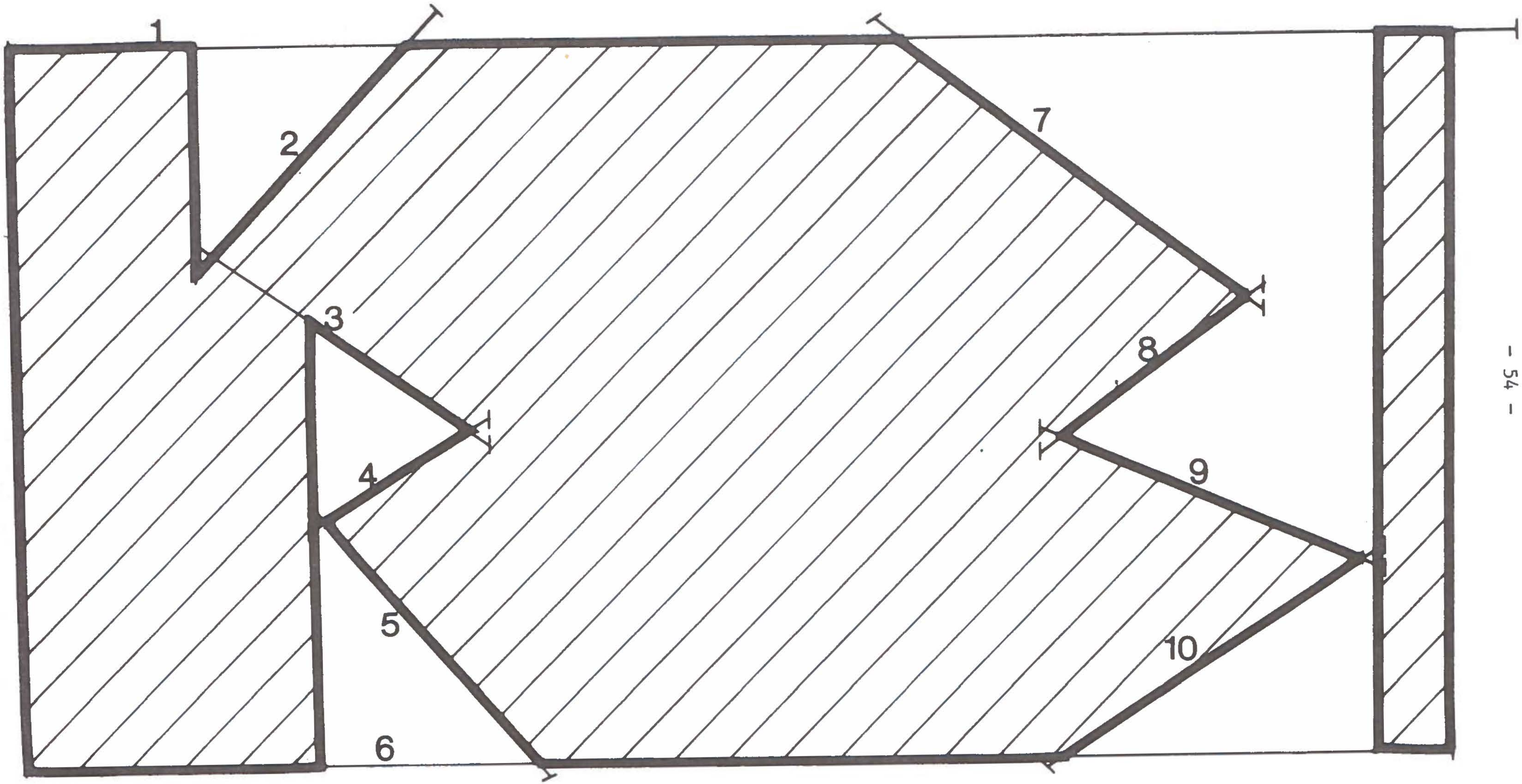


Fig. 9 c)



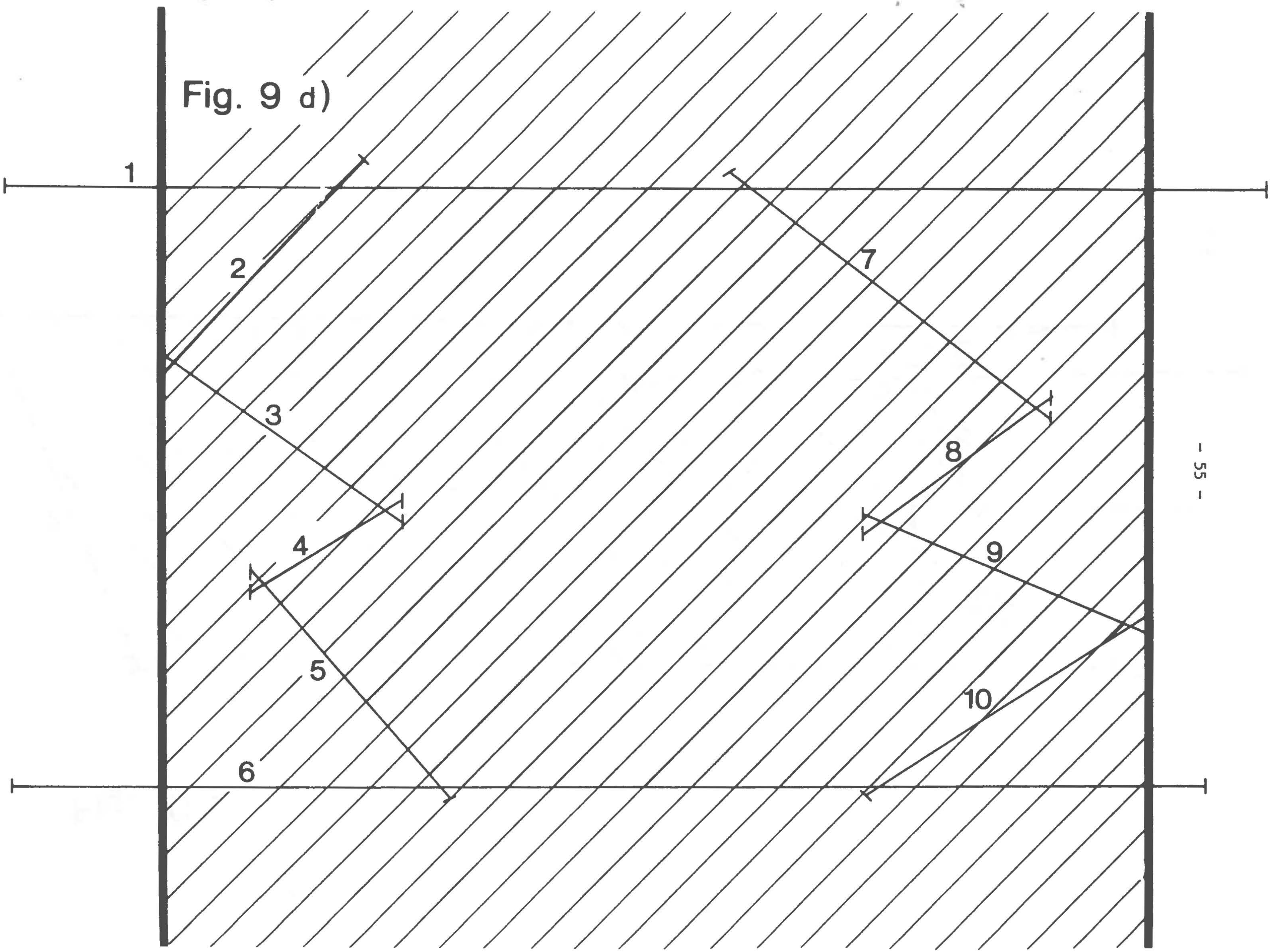


Fig. 9 d)

Fig. 10 a)

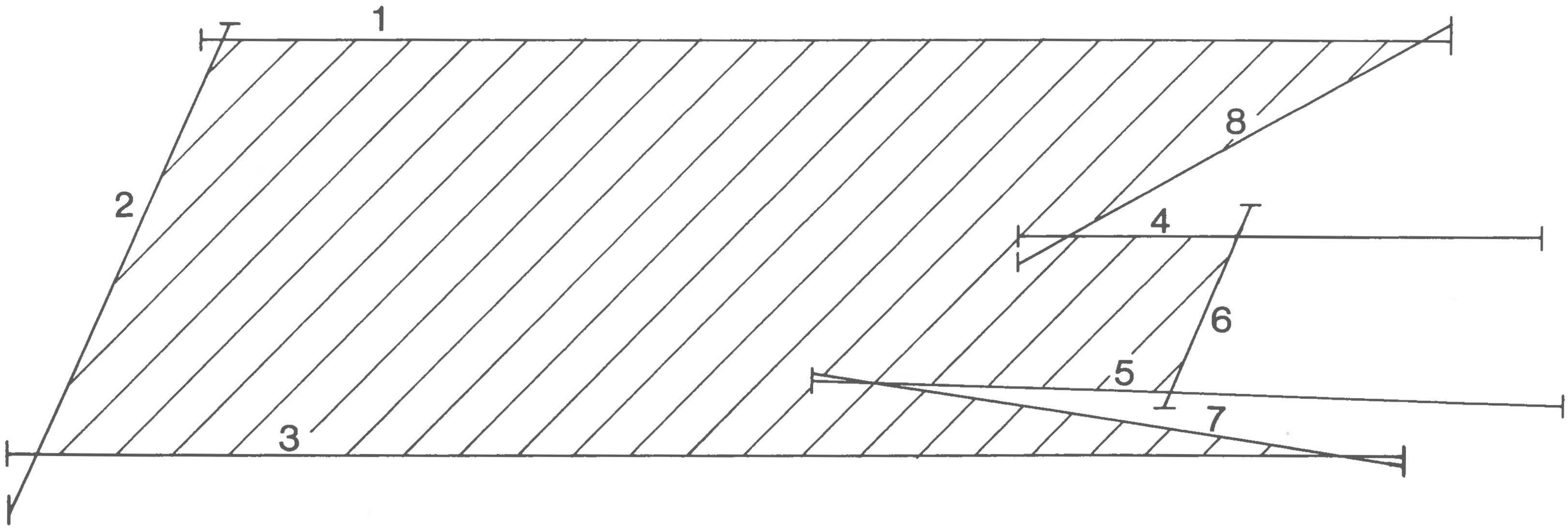


Fig. 10 b)